Lesson 12: Properties of Logarithms

Student Outcomes

- Students justify properties of logarithms using the definition and properties already developed.

Lesson Notes

In this lesson, students work exclusively with logarithms base 10; generalization of these results to a generic base b will occur in the next lesson. The opening of this lesson, which echoes homework from Lesson 11, is meant to launch a consideration of some properties of the common logarithm function. The centerpiece of the lesson is the demonstration of six basic properties of logarithms theoretically using the properties of exponents instead of numerical approximation as has been done in prior lessons. In the Problem Set, students will apply these properties to calculating logarithms, rewriting logarithmic expressions, and solving exponential equations base 10 (**A-SSE.A.2**, **F-LE.A.4**).

Classwork

Opening Exercise (5 minutes)

Students should work in groups of two or three on this exercise. These exercises serve to remind students of the "most important property" of logarithms and prepare them for justifying the properties later in the lesson. Verify that students are breaking up the logarithm, using the property, and evaluating the logarithm at known values (e.g., 0.1, 10, 100).

Scaffolding:

- Ask students who are having trouble with any part of this exercise, "How is the number in parentheses related to 2?" Follow that with the question, "So how might you find its logarithm given that you know $\log(2)$?"
- Ask students to factor each number into powers of 10 and factors of 2 before splitting the factors using $\log(xy) = \log(x) + \log(y)$. Students still struggling can be given additional products to break down before finding the approximations of their logarithms.

$$4 = 2 \cdot 2$$
$$40 = 10^1 \cdot 2 \cdot 2$$
$$0.4 = 10^{-1} \cdot 2 \cdot 2$$
$$400 = 10^2 \cdot 2 \cdot 2$$
$$0.04 = 10^{-2} \cdot 2 \cdot 2$$

- Advanced students may be challenged with a more general version of part (c):

$$\log(2^k).$$

This can be explored by having students find $\log(2^5)$, $\log(2^6)$, etc.

Opening Exercise

Use the approximation $\log(2) \approx 0.3010$ to approximate the values of each of the following logarithmic expressions.

a. $\log(20)$

$$\begin{aligned}\log(20) &= \log(10 \cdot 2)\\ &= \log(10) + \log(2)\\ &\approx 1 + 0.301\\ &\approx 1.301\end{aligned}$$

b. $\log(0.2)$

$$\begin{aligned}\log(0.2) &= \log(0.1 \cdot 2)\\ &= \log(0.1) + \log(2)\\ &\approx -1 + 0.3010\\ &\approx -0.6990\end{aligned}$$

c. $\log(2^4)$

$$\log(2^4) = \log(2\cdot 2\cdot 2\cdot 2)$$
$$= \log(2\cdot 2) + \log(2\cdot 2)$$
$$= \log(2) + \log(2) + \log(2) + \log(2)$$
$$\approx 4\cdot(0.3010)$$
$$\approx 1.2040$$

Discussion (4 minutes)

Discuss the properties of logarithms used in the Opening Exercises.

- In all three parts of the Opening Exercise, we used the important property $\log(xy) = \log(x) + \log(y)$.
- What are some other properties we used?
  - *We also used* $\log(10) = 1$ *and* $\log(0.1) = -1$.
- It can be helpful to look further at properties of expressions involving logarithms.

Example (6 minutes)

Recall that, by definition, $L = \log(x)$ means $10^L = x$. Consider some possible values of x and L, noting that x cannot be a negative number. What is L …

- When $x = 1$?
  - $L = 0$
- When $x = 0$?
  - *The logarithm* L *is not defined. There is no exponent of* 10 *that yields a value of* 0.
- When $x = 10^9$?
  - $L = 9$
- When $x = 10^n$?
  - $L = n$
- When $x = \sqrt[3]{10}$?
  - $L = \frac{1}{3}$

Exercises 1–6 (15 minutes)

Students should work in groups of two or three on each exercise. The first three should be straightforward in view of the definition of base 10 logarithms. Exercise 4 may look somewhat odd, but it, too, follows directly from the definition. Exercises 5 and 6 are more difficult, which is why the hints are supplied. When all properties have been established, groups might be asked to show their explanations to the rest of the class as time permits.

MP.3

Exercises

For Exercises 1–6, explain why each statement below is a property of base 10 logarithms.

1. **Property 1: $\log(1) = 0$.**

   Because $L = \log(x)$ means $10^L = x$, then when $x = 1$, $L = 0$.

2. **Property 2: $\log(10) = 1$.**

   Because $L = \log(x)$ means $10^L = x$, then when $x = 10$, $L = 1$.

3. **Property 3: For all real numbers r, $\log(10^r) = r$.**

   Because $L = \log(x)$ means $10^L = x$, then when $x = 10^r$, $L = r$.

4. **Property 4: For any $x > 0$, $10^{\log(x)} = x$.**

   Because $L = \log(x)$ means $10^L = x$, then $x = 10^{\log(x)}$.

5. **Property 5: For any positive real numbers x and y, $\log(x \cdot y) = \log(x) + \log(y)$.**
   Hint: Use an exponent rule as well as property 4.

   By the rule $a^b \cdot a^c = a^{b+c}$, $10^{\log(x)} \cdot 10^{\log(y)} = 10^{\log(x)+\log(y)}$.

   By property 4, $10^{\log(x)} \cdot 10^{\log(y)} = x \cdot y$.

   Therefore, $x \cdot y = 10^{\log(x)+\log(y)}$. Again, by property 4, $x \cdot y = 10^{\log(x \cdot y)}$.

   Then, $10^{\log(x \cdot y)} = 10^{\log(x)+\log(y)}$; so, the exponents must be equal, and $\log(x \cdot y) = \log(x) + \log(y)$.

6. **Property 6: For any positive real number x and any real number r, $\log(x^r) = r \cdot \log(x)$.**
   Hint: Again, use an exponent rule as well as property 4.

   By the rule $(a^b)^c = a^{bc}$, $10^{k\log(x)} = \left(10^{\log(x)}\right)^k$.

   By property 4, $\left(10^{\log(x)}\right)^r = x^r$.

   Therefore, $x^r = 10^{r\log(x)}$. Again, by property 4, $x^r = 10^{\log(x^r)}$.

   Then, $10^{\log(x^r)} = 10^{r\log(x)}$; so, the exponents must be equal, and $\log(x^r) = r \cdot \log(x)$.

Scaffolding:

Establishing the logarithmic properties relies on the exponential laws. Make sure that students have access to the exponential laws either through a poster displayed in the classroom or through notes in their notebooks.

Scaffolding:

Students in groups that struggle with Exercises 3–6 should be encouraged to check the property with numerical values for k, x, m, and n. The check may suggest an explanation.

Exercises 7–10 (8 minutes)

These exercises bridge the gap between the abstract properties of logarithms and computational problems like those in the Problem Set. Allow students to work alone, in pairs, or in small groups as you see fit. Circulate to ensure that students are applying the properties correctly. Calculators are not needed for these exercises and should not be used. In Exercises 9 and 10, students need to know that the logarithm is well-defined; that is, for positive real numbers X and Y, if $X = Y$, then $\log(X) = \log(Y)$. This is why we can “take the log of both sides” of an equation in order to bring down an exponent and solve the equation. In these last two exercises, students need to choose an appropriate base for the logarithm to apply to solve the equation. Any logarithm will work to solve the equations if applied properly, so students may find equivalent answers that appear to be different from those listed here.

7. **Apply properties of logarithms to rewrite the following expressions as a single logarithm or number.**

a. $\frac{1}{2}\log(25) + \log(4)$

$\log(5) + \log(4) = \log(20)$

b. $\frac{1}{3}\log(8) + \log(16)$

$\log(2) + \log(2^4) = \log(32)$

c. $3\log(5) + \log(0.8)$

$\log(125) + \log(0.8) = \log(100) = 2$

8. **Apply properties of logarithms to rewrite each expression as a sum of terms involving numbers, $\log(x)$, and $\log(y)$.**

a. $\log(3x^2y^5)$

$\log(3) + 2\log(x) + 5\log(y)$

b. $\log\left(\sqrt{x^7y^3}\right)$

$\frac{7}{2}\log(x) + \frac{3}{2}\log(y)$

9. **In mathematical terminology, logarithms are well defined because if $X = Y$, then $\log(X) = \log(Y)$ for $X, Y > 0$. This means that if you want to solve an equation involving exponents, you can apply a logarithm to both sides of the equation, just as you can take the square root of both sides when solving a quadratic equation. You do need to be careful not to take the logarithm of a negative number or zero.**

Use the property stated above to solve the following equations.

a. $10^{10x} = 100$

$\log(10^{10x}) = \log(100)$
$10x = 2$
$x = \frac{1}{5}$

b. $10^{x-1} = \frac{1}{10^{x+1}}$

$\log(10^{x-1}) = -\log(10^{x+1})$
$x - 1 = -(x+1)$
$2x = 0$
$x = 0$

c. $100^{2x} = 10^{3x-1}$

$\log(100^{2x}) = \log(10^{3x-1})$
$2x\log(100) = (3x-1)$
$4x = 3x - 1$
$x = -1$

10. **Solve the following equations.**

a. $10^x = 2^7$

$$\log(10^x) = \log(2^7)$$
$$x = 7\log(2)$$

b. $10^{x^2+1} = 15$

$$\log\left(10^{x^2+1}\right) = \log(15)$$
$$x^2 + 1 = \log(15)$$
$$x = \pm\sqrt{\log(15) - 1}$$

c. $4^x = 5^3$

$$\log(4^x) = \log(5^3)$$
$$x\log(4) = 3\log(5)$$
$$x = \frac{3\log(5)}{\log(4)}$$

Closing (2 minutes)

Point out that for each property 1–6, we have established that the property holds, so we can use these properties in our future work with logarithms. The Lesson Summary might be posted in the classroom for at least the rest of the module. The Exit Ticket asks the students to show that properties 7 and 8 hold.

Lesson Summary

We have established the following properties for base 10 logarithms, where x and y are positive real numbers and r is any real number:

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

Additional properties not yet established are the following:

1. $\log\left(\frac{1}{x}\right) = -\log(x)$
2. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

Also, logarithms are well defined, meaning that for $X, Y > 0$, if $X = Y$, then $\log(X) = \log(Y)$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: Properties of Logarithms

Exit Ticket

1. State as many of the six properties of logarithms as you can.

2. Use the properties of logarithms to show that $\log\left(\frac{1}{x}\right) = -\log(x)$ for all $x > 0$.

3. Use the properties of logarithms to show that $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$ for $x > 0$ and $y > 0$.

Exit Ticket Sample Solutions

1. **State as many of the six properties of logarithms as you can.**

   $\log(1) = 0$

   $\log(10) = 1$

   $\log(10^r) = r$

   $10^{\log(x)} = x$

   $\log(x \cdot y) = \log(x) + \log(y)$

   $\log(x^r) = r \cdot \log(x)$

2. **Use the properties of logarithms to show that $\log\left(\frac{1}{x}\right) = -\log(x)$ for $x > 0$.**

   By property 6, $\log(x^k) = k \cdot \log(x)$.

   Let $k = -1$, *then for* $x > 0$, $\log(x^{-1}) = (-1) \cdot \log(x)$, *which is equivalent to* $\log\left(\frac{1}{x}\right) = -\log(x)$.

   Thus, for any $x > 0$, $\log\left(\frac{1}{x}\right) = -\log(x)$.

3. **Use the properties of logarithms to show that $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$ for $x > 0$ and $y > 0$.**

   By property 5, $\log(x \cdot y) = \log(x) + \log(y)$.

   By Problem 2 above, for $y > 0$, $\log(y^{-1}) = (-1) \cdot \log(y)$.

   Therefore,

$$\begin{aligned}\log\left(\frac{x}{y}\right) &= \log(x) + \log\left(\frac{1}{y}\right)\\ &= \log(x) + (-1)\log(y)\\ &= \log(x) - \log(y).\end{aligned}$$

   Thus, for any $x > 0$ *and* $y > 0$, $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$.

Problem Set Sample Solutions

Problems 1–7 give students an opportunity to practice using the properties they have established in this lesson, and in the remaining problems, students apply base 10 logarithms to solve simple exponential equations.

1. **Use the approximate logarithm values below to estimate each of the following logarithms. Indicate which properties you used.**

   $\log(2) = 0.3010 \qquad \log(3) = 0.4771$

   $\log(5) = 0.6990 \qquad \log(7) = 0.8451$

   a. $\log(6)$

      Using property 5,

      $\log(6) = \log(3) + \log(2) \approx 0.7781$.

b. $\log(15)$

Using property 5,

$\log(15) = \log(3) + \log(5) \approx 1.1761.$

c. $\log(12)$

Using properties 5 and 6,

$\log(12) = \log(3) + \log(2^2) = \log(3) + 2\log(2) \approx 1.0791.$

d. $\log(10^7)$

Using property 3,

$\log(10^7) = 7.$

e. $\log\left(\frac{1}{5}\right)$

Using property 7,

$\log\left(\frac{1}{5}\right) = -\log(5) \approx -0.6990.$

f. $\log\left(\frac{3}{7}\right)$

Using property 8,

$\log\left(\frac{3}{7}\right) = \log(3) - \log(7) \approx -0.368.$

g. $\log(\sqrt[4]{2})$

Using property 6,

$\log(\sqrt[4]{2}) = \log\left(2^{\frac{1}{4}}\right) = \frac{1}{4}\log(2) \approx 0.0753.$

2. Let $\log(X) = r$, $\log(Y) = s$, and $\log(Z) = t$. Express each of the following in terms of r, s, and t.

a. $\log\left(\frac{X}{Y}\right)$

$r - s$

b. $\log(YZ)$

$s + t$

c. $\log(X^r)$

r^2

d. $\log(\sqrt[3]{Z})$

$\frac{t}{3}$

e. $\log\left(\sqrt[4]{\frac{Y}{Z}}\right)$

$\frac{s-t}{4}$

f. $\log(XY^2Z^3)$

$r + 2s + 3t$

3. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.

a. $\log\left(\frac{13}{5}\right) + \log\left(\frac{5}{4}\right)$

$\log\left(\frac{13}{4}\right)$

b. $\log\left(\frac{5}{6}\right) - \log\left(\frac{2}{3}\right)$

$\log\left(\frac{5}{4}\right)$

c. $\frac{1}{2}\log(16) + \log(3) + \log\left(\frac{1}{4}\right)$

$\log(3)$

4. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.

a. $\log(\sqrt{x}) + \frac{1}{2}\log\left(\frac{1}{x}\right) + 2\log(x)$

$\log(x^2)$

b. $\log(\sqrt[5]{x}) + \log(\sqrt[5]{x^4})$

$\log(x)$

c. $\log(x) + 2\log(y) - \frac{1}{2}\log(z)$

$\log\left(\frac{xy^2}{\sqrt{z}}\right)$

d. $\frac{1}{3}(\log(x) - 3\log(y) + \log(z))$

$\log\left(\sqrt[3]{\frac{xz}{y^3}}\right)$

e. $2(\log(x) - \log(3y)) + 3(\log(z) - 2\log(x))$

$\log\left(\left(\frac{x}{3y}\right)^2\right) + \log\left(\left(\frac{z}{x^2}\right)^3\right) = \log\left(\frac{z^3}{9y^2x^4}\right)$

5. Use properties of logarithms to rewrite the following expressions in an equivalent form containing only $\log(x)$, $\log(y)$, $\log(z)$, and numbers.

a. $\log\left(\frac{3x^2y^4}{\sqrt{z}}\right)$

$\log(3) + 2\log(x) + 4\log(y) - \frac{1}{2}\log(z)$

b. $\log\left(\frac{42\sqrt[3]{xy^7}}{x^2z}\right)$

$\log(42) - \frac{5}{3}\log(x) + \frac{7}{3}\log(y) - \log(z)$

c. $\log\left(\frac{100x^2}{y^3}\right)$

$2 + 2\log(x) - 3\log(y)$

d. $\log\left(\sqrt{\frac{x^3y^2}{10z}}\right)$

$\frac{1}{2}(3\log(x) + 2\log(y) - 1 - \log(z))$

e. $\log\left(\frac{1}{10x^2z}\right)$

$-1 - 2\log(x) - \log(z)$

6. Express $\log\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\right)$ as a single logarithm for positive numbers x.

$$\begin{aligned}\log\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\right) &= \log\left(\frac{1}{x(x+1)}\right) + \log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\\ &= -\log(x(x+1)) - \log(x) + \log(x+1)\\ &= -\log(x) - \log(x+1) - \log(x) + \log(x+1)\\ &= -2\log(x)\end{aligned}$$

7. Show that $\log(x + \sqrt{x^2-1}) + \log(x - \sqrt{x^2-1}) = 0$ for $x \geq 1$.

$$\begin{aligned}\log\left(x + \sqrt{x^2-1}\right) + \log\left(x - \sqrt{x^2-1}\right) &= \log\left(\left(x + \sqrt{x^2-1}\right)\left(x - \sqrt{x^2-1}\right)\right)\\ &= \log\left(x^2 - \left(\sqrt{x^2-1}\right)^2\right)\\ &= \log(x^2 - x^2 + 1)\\ &= \log(1)\\ &= 0\end{aligned}$$

8. If $xy = 10^{3.67}$, find the value of $\log(x) + \log(y)$.

$$\begin{aligned}xy &= 10^{3.67}\\ 3.67 &= \log(xy)\\ \log(xy) &= 3.67\\ \log(x) + \log(y) &= 3.67\end{aligned}$$

9. Solve the following exponential equations by taking the logarithm base 10 of both sides. Leave your answers stated in terms of logarithmic expressions.

a. $10^{x^2} = 320$

$$\begin{aligned}\log\left(10^{x^2}\right) &= \log(320)\\ x^2 &= \log(320)\\ x &= \pm\sqrt{\log(320)}\end{aligned}$$

b. $10^{\frac{x}{8}} = 300$

$$\log\left(10^{\frac{x}{8}}\right) = \log(300)$$
$$\frac{x}{8} = \log(10^2 \cdot 3)$$
$$\frac{x}{8} = 2 + \log(3)$$
$$x = 16 + 8\log(3)$$

c. $10^{3x} = 400$

$$\log(10^{3x}) = \log(400)$$
$$3x \cdot \log(10) = \log(10^2 \cdot 4)$$
$$3x \cdot 1 = 2 + \log(4)$$
$$x = \frac{1}{3}(2 + \log(4))$$

d. $5^{2x} = 200$

$$\log(5^{2x}) = \log(200)$$
$$2x \cdot \log(5) = \log(100) + \log(2)$$
$$2x = \frac{2 + \log(2)}{\log(5)}$$
$$x = \frac{2 + \log(2)}{2\log(5)}$$

e. $3^{x} = 7^{-3x+2}$

$$\log(3^x) = \log\left(7^{-3x+2}\right)$$
$$x\log(3) = (-3x + 2)\log(7)$$
$$x\log(3) + 3x\log(7) = 2\log(7)$$
$$x\big(\log(3) + 3\log(7)\big) = 2\log(7)$$
$$x = \frac{2\log(7)}{\log(3) + 3\log(7)} = \frac{\log(49)}{\log(3) + \log(343)} = \frac{\log(49)}{\log(1029)}$$

(Any of the three equivalent forms given above are acceptable answers.)

10. **Solve the following exponential equations.**

a. $10^x = 3$

$x = \log(3)$

b. $10^y = 30$

$y = \log(30)$

c. $10^z = 300$

$z = \log(300)$

d. Use the properties of logarithms to justify why x, y, and z form an arithmetic sequence whose constant difference is 1.

Since $y = \log(30)$, $y = \log(10 \cdot 3) = 1 + \log(10) = 1 + x$.
Similarly, $z = 2 + \log(3) = 2 + x$.

Thus, the sequence x, y, z is the sequence $\log(3)$, $1 + \log(3)$, $2 + \log(3)$, and these numbers form an arithmetic sequence whose first term is $\log(3)$ with a constant difference of 1.

11. Without using a calculator, explain why the solution to each equation must be a real number between 1 and 2.

a. $11^x = 12$

12 is greater than 11^1 and less than 11^2, so the solution is between 1 and 2.

b. $21^x = 30$

30 is greater than 21^1 and less than 21^2, so the solution is between 1 and 2.

c. $100^x = 2000$

$100^2 = 10000$, and 2000 is less than that, so the solution is between 1 and 2.

d. $\left(\frac{1}{11}\right)^x = 0.01$

$\frac{1}{100}$ is between $\frac{1}{11}$ and $\frac{1}{121}$, so the solution is between 1 and 2.

e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$

$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, and $\frac{1}{2}$ is between $\frac{2}{3}$ and $\frac{4}{9}$, so the solution is between 1 and 2.

f. $99^x = 9000$

$99^2 = 9801$. Since 9000 is less than 9801 and greater than 99, the solution is between 1 and 2.

12. Express the exact solution to each equation as a base 10 logarithm. Use a calculator to approximate the solution to the nearest 1000^{th}.

a. $11^x = 12$

$$\log(11^x) = \log(12)$$
$$x\log(11) = \log(12)$$
$$x = \frac{\log(12)}{\log(11)}$$
$$x \approx 1.036$$

b. $21^x = 30$

$$x = \frac{\log(30)}{\log(21)}$$
$$x \approx 1.117$$

c. $100^x = 2000$

$$x = \frac{\log(2000)}{\log(100)}$$
$$x \approx 1.651$$

d. $\left(\frac{1}{11}\right)^x = 0.01$

$$x = -\frac{2}{\log\left(\frac{1}{11}\right)}$$
$$x \approx 1.921$$

e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$

$$x = \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{2}{3}\right)}$$
$$x \approx 1.710$$

f. $99^x = 9000$

$$x = \frac{\log(9000)}{\log(99)}$$
$$x \approx 1.981$$

13. Show that the value of x that satisfies the equation $10^x = 3 \cdot 10^n$ is $\log(3) + n$.

Substituting $x = \log(3) + n$ into 10^x and using properties of exponents and logarithms gives

$$\begin{aligned} 10^x &= 10^{\log(3)+n} \\ &= 10^{\log(3)}10^n \\ &= 3 \cdot 10^n. \end{aligned}$$

Thus, $x = \log(3) + n$ is a solution to the equations $10^x = 3 \cdot 10^n$.

14. Solve each equation. If there is no solution, explain why.

a. $3 \cdot 5^x = 21$

$$\begin{aligned} 5^x &= 7 \\ \log(5^x) &= \log(7) \\ x\log(5) &= \log(7) \\ x &= \frac{\log(7)}{\log(5)} \end{aligned}$$

b. $10^{x-3} = 25$

$$\begin{aligned} \log(10^{x-3}) &= \log(25) \\ x &= 3 + \log(25) \end{aligned}$$

c. $10^x + 10^{x+1} = 11$

$$10^x(1+10) = 11$$
$$10^x = 1$$
$$x = 0$$

d. $8 - 2^x = 10$

$$-2^x = 2$$
$$2^x = -2$$

There is no solution because 2^x is always positive for all real x

15. Solve the following equation for n: $A = P(1+r)^n$.

$$A = P(1+r)^n$$
$$\log(A) = \log[(P(1+r)^n]$$
$$\log(A) = \log(P) + \log[(1+r)^n]$$
$$\log(A) - \log(P) = n\log(1+r)$$
$$n = \frac{\log(A) - \log(P)}{\log(1+r)}$$
$$n = \frac{\log\left(\frac{A}{P}\right)}{\log(1+r)}$$

The remaining questions establish a property for the logarithm of a sum. Although this is an application of the logarithm of a product, the formula does have some applications in information theory and can help with the calculations necessary to use tables of logarithms, which will be explored further in Lesson 15.

16. In this exercise, we will establish a formula for the logarithm of a sum. Let $L = \log(x+y)$, where $x, y > 0$.

a. Show $\log(x) + \log\left(1 + \frac{y}{x}\right) = L$. State as a property of logarithms after showing this is a true statement.

$$\log(x) + \log\left(1+\frac{y}{x}\right) = \log\left(x\left(1+\frac{y}{x}\right)\right)$$
$$= \log\left(x + \frac{xy}{x}\right)$$
$$= \log(x+y)$$
$$= L$$

Therefore, for $x, y > 0$, $\log(x+y) = \log(x) + \log\left(1+\frac{y}{x}\right)$.

b. Use part (a) and the fact that $\log(100) = 2$ to rewrite $\log(365)$ as a sum.

$$\log(365) = \log(100 + 265)$$
$$= \log(100) + \log\left(1 + \frac{265}{100}\right)$$
$$= \log(100) + \log(3.65)$$
$$= 2 + \log(3.65)$$

c. **Rewrite 365 in scientific notation, and use properties of logarithms to express $\log(365)$ as a sum of an integer and a logarithm of a number between 0 and 10.**

$$\begin{aligned} 365 &= 3.65 \times 10^2 \\ \log(365) &= \log(3.65 \times 10^2) \\ &= \log(3.65) + \log(10^2) \\ &= 2 + \log(3.65) \end{aligned}$$

d. **What do you notice about your answers to (b) and (c)?**

Separating 365 into $100 + 265$ and using the formula for the logarithm of a sum is the same as writing 365 in scientific notation and using formula for the logarithm of a product.

e. **Find two integers that are upper and lower estimates of $\log(365)$.**

Since $1 < 3.65 < 10$, we know that $0 < \log(3.65) < 1$. This tells us that $2 < 2 + \log(3.65) < 3$, so $2 < \log(365) < 3$.