Polynomial and Rational Functions

2

In Mathematics

Functions defined by polynomial expressions are called polynomial functions, and functions defined by rational expressions are called rational functions.

In Real Life

Polynomial and rational functions are often used to model real-life phenomena. For instance, you can model the per capita cigarette consumption in the United States with a polynomial function. You can use the model to determine whether the addition of cigarette warnings affected consumption. (See Exercise 85, page 134.)

Michael Newman/PhotoEdit

IN CAREERS

There are many careers that use polynomial and rational functions. Several are listed below.

- Architect
  Exercise 82, page 134
- Forester
  Exercise 103, page 148
- Chemist
  Example 80, page 192
- Safety Engineer
  Exercise 78, page 203

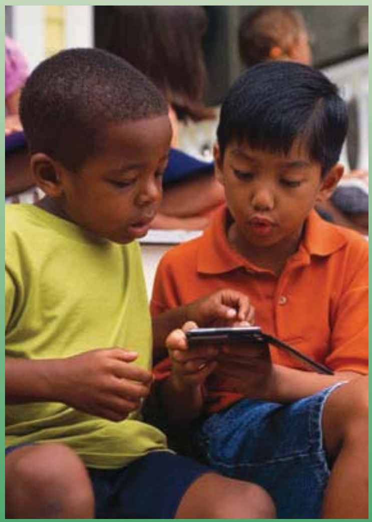
2.1 Quadratic Functions and Models

What you should learn

- Analyze graphs of quadratic functions.
- Write quadratic functions in standard form and use the results to sketch graphs of functions.
- Find minimum and maximum values of quadratic functions in real-life applications.

Why you should learn it

Quadratic functions can be used to model data to analyze consumer behavior. For instance, in Exercise 79 on page 134, you will use a quadratic function to model the revenue earned from manufacturing handheld video games.



The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions. In Section 1.6, you were introduced to the following basic functions.

$f(x) = ax + b$ Linear function

$f(x) = c$ Constant function

$f(x) = x^2$ Squaring function

These functions are examples of **polynomial functions.**

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n.**

Polynomial functions are classified by degree. For instance, a constant function $f(x) = c$ with $c \neq 0$ has degree 0, and a linear function $f(x) = ax + b$ with $a \neq 0$ has degree 1. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

For instance, each of the following functions is a quadratic function.

$$f(x) = x^2 + 6x + 2$$

$$g(x) = 2(x + 1)^2 - 3$$

$$h(x) = 9 + \tfrac{1}{4}x^2$$

$$k(x) = -3x^2 + 4$$

$$m(x) = (x - 2)(x + 1)$$

Note that the squaring function is a simple quadratic function that has degree 2.

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function given by

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a **quadratic function.**

The graph of a quadratic function is a special type of "U"-shaped curve called a **parabola.** Parabolas occur in many real-life applications—especially those involving reflective properties of satellite dishes and flashlight reflectors. You will study these properties in Section 10.2.

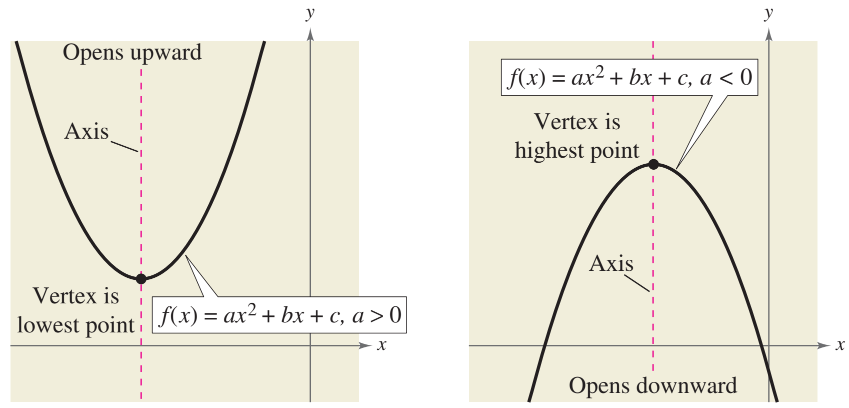
All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola, as shown in Figure 2.1. If the leading coefficient is positive, the graph of

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward. If the leading coefficient is negative, the graph of

$$f(x) = ax^2 + bx + c$$

is a parabola that opens downward.



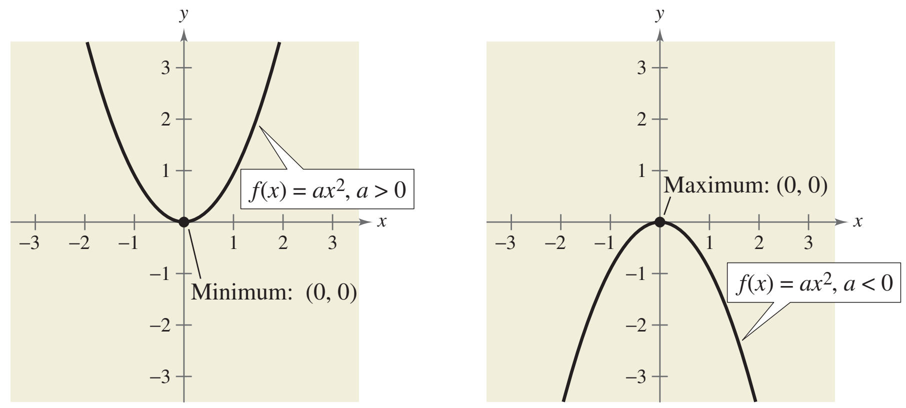
Leading coefficient is positive. Leading coefficient is negative.

FIGURE 2.1

The simplest type of quadratic function is

$$f(x) = ax^2.$$

Its graph is a parabola whose vertex is (0, 0). If $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and if $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown in Figure 2.2.



Leading coefficient is positive. Leading coefficient is negative.

FIGURE 2.2

When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 1.7.

Example 1 Sketching Graphs of Quadratic Functions

a. Compare the graphs of $y = x^2$ and $f(x) = \frac{1}{3}x^2$.

b. Compare the graphs of $y = x^2$ and $g(x) = 2x^2$.

Algebra Help

You can review the techniques for shifting, reflecting, and stretching graphs in Section 1.7.

Solution

a. Compared with $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ "shrinks" by a factor of $\frac{1}{3}$, creating the broader parabola shown in Figure 2.3.

b. Compared with $y = x^2$, each output of $g(x) = 2x^2$ "stretches" by a factor of 2, creating the narrower parabola shown in Figure 2.4.

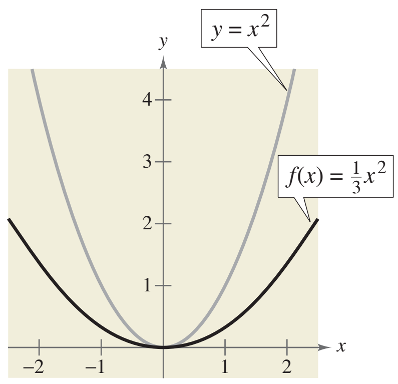


FIGURE 2.3

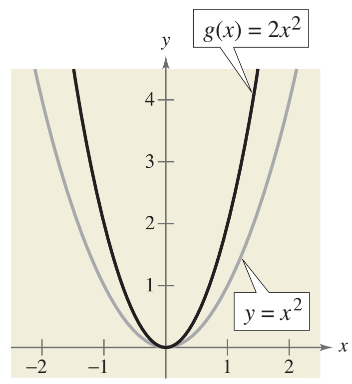
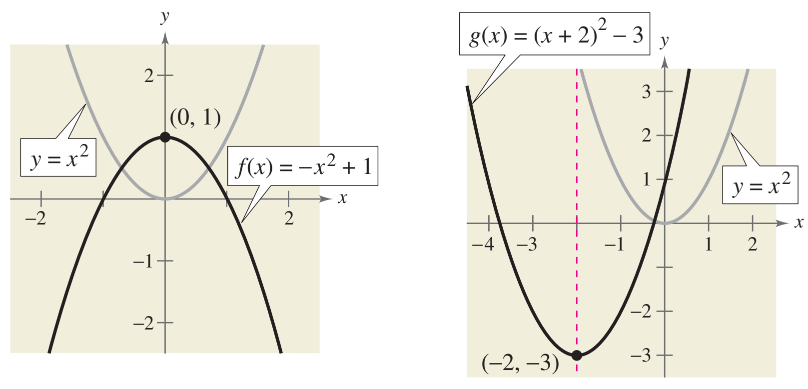


FIGURE 2.4

CHECKPoint Now try Exercise 13.

In Example 1, note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. If $|a|$ is small, the parabola opens more widely than if $|a|$ is large.

Recall from Section 1.7 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = f(-x)$, and $y = -f(x)$ are rigid transformations of the graph of $y = f(x)$. For instance, in Figure 2.5, notice how the graph of $y = x^2$ can be transformed to produce the graphs of $f(x) = -x^2 + 1$ and $g(x) = (x + 2)^2 - 3$.



Reflection in x-axis followed by an upward shift of one unit

Left shift of two units followed by a downward shift of three units

FIGURE 2.5

The Standard Form of a Quadratic Function

The **standard form** of a quadratic function is $f(x) = a(x - h)^2 + k$. This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

> **Study Tip**
>
> The standard form of a quadratic function identifies four basic transformations of the graph of $y = x^2$.
>
> **a.** The factor $|a|$ produces a vertical stretch or shrink.
>
> **b.** If $a < 0$, the graph is reflected in the x-axis.
>
> **c.** The factor $(x - h)^2$ represents a horizontal shift of h units.
>
> **d.** The term k represents a vertical shift of k units.

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

To graph a parabola, it is helpful to begin by writing the quadratic function in standard form using the process of completing the square, as illustrated in Example 2. In this example, notice that when completing the square, you *add and subtract* the square of half the coefficient of x within the parentheses instead of adding the value to each side of the equation as is done in Appendix A.5.

Example 2 Graphing a Parabola in Standard Form

Sketch the graph of $f(x) = 2x^2 + 8x + 7$ and identify the vertex and the axis of the parabola.

Solution

Begin by writing the quadratic function in standard form. Notice that the first step in completing the square is to factor out any coefficient of x^2 that is not 1.

> **Algebra Help**
>
> You can review the techniques for completing the square in Appendix A.5.

$$f(x) = 2x^2 + 8x + 7 \qquad \text{Write original function.}$$

$$= 2(x^2 + 4x) + 7 \qquad \text{Factor 2 out of } x\text{-terms.}$$

$$= 2(x^2 + 4x + \underbrace{4}_{(4/2)^2} - 4) + 7 \qquad \text{Add and subtract 4 within parentheses.}$$

After adding and subtracting 4 within the parentheses, you must now regroup the terms to form a perfect square trinomial. The -4 can be removed from inside the parentheses; however, because of the 2 outside of the parentheses, you must multiply -4 by 2, as shown below.

$$f(x) = 2(x^2 + 4x + 4) - 2(4) + 7 \qquad \text{Regroup terms.}$$

$$= 2(x^2 + 4x + 4) - 8 + 7 \qquad \text{Simplify.}$$

$$= 2(x + 2)^2 - 1 \qquad \text{Write in standard form.}$$

From this form, you can see that the graph of f is a parabola that opens upward and has its vertex at $(-2, -1)$. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in Figure 2.6. In the figure, you can see that the axis of the parabola is the vertical line through the vertex, $x = -2$.

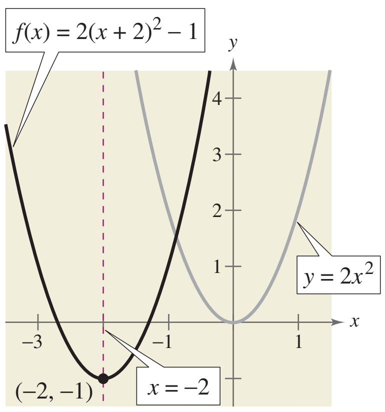


FIGURE 2.6

CHECKPoint Now try Exercise 19.

Algebra Help

You can review the techniques for using the Quadratic Formula in Appendix A.5.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation $ax^2 + bx + c = 0$. If $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts. Remember, however, that a parabola may not have x-intercepts.

Example 3 Finding the Vertex and x-Intercepts of a Parabola

Sketch the graph of $f(x) = -x^2 + 6x - 8$ and identify the vertex and x-intercepts.

Solution

$f(x) = -x^2 + 6x - 8$ — Write original function.

$= -(x^2 - 6x) - 8$ — Factor -1 out of x-terms.

$= -(x^2 - 6x + 9 - 9) - 8$ — Add and subtract 9 within parentheses.

$(-6/2)^2$

$= -(x^2 - 6x + 9) - (-9) - 8$ — Regroup terms.

$= -(x - 3)^2 + 1$ — Write in standard form.

From this form, you can see that f is a parabola that opens downward with vertex $(3, 1)$. The x-intercepts of the graph are determined as follows.

$-(x^2 - 6x + 8) = 0$ — Factor out -1.

$-(x - 2)(x - 4) = 0$ — Factor.

$x - 2 = 0 \Rightarrow x = 2$ — Set 1st factor equal to 0.

$x - 4 = 0 \Rightarrow x = 4$ — Set 2nd factor equal to 0.

So, the x-intercepts are $(2, 0)$ and $(4, 0)$, as shown in Figure 2.7.

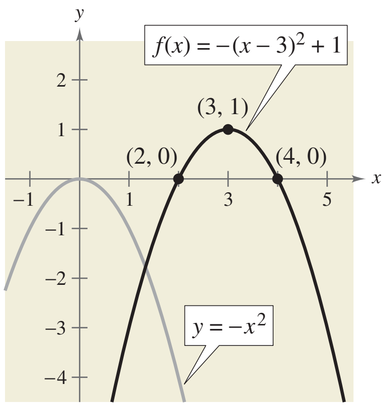


FIGURE 2.7

CHECKPoint Now try Exercise 25.

Example 4 Writing the Equation of a Parabola

Write the standard form of the equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(3, -6)$.

Solution

Because the vertex of the parabola is at $(h, k) = (1, 2)$, the equation has the form

$f(x) = a(x - 1)^2 + 2.$ — Substitute for h and k in standard form.

Because the parabola passes through the point $(3, -6)$, it follows that $f(3) = -6$. So,

$f(x) = a(x - 1)^2 + 2$ — Write in standard form.

$-6 = a(3 - 1)^2 + 2$ — Substitute 3 for x and -6 for $f(x)$.

$-6 = 4a + 2$ — Simplify.

$-8 = 4a$ — Subtract 2 from each side.

$-2 = a.$ — Divide each side by 4.

The equation in standard form is $f(x) = -2(x - 1)^2 + 2$. The graph of f is shown in Figure 2.8.

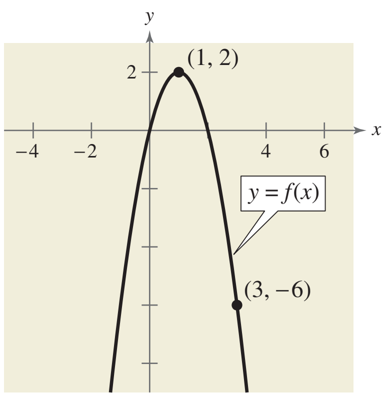


FIGURE 2.8

CHECKPoint Now try Exercise 47.

Finding Minimum and Maximum Values

Many applications involve finding the maximum or minimum value of a quadratic function. By completing the square of the quadratic function $f(x) = ax^2 + bx + c$, you can rewrite the function in standard form (see Exercise 95).

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Standard form}$$

So, the vertex of the graph of f is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, which implies the following.

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. If $a > 0$, f has a *minimum* at $x = -\frac{b}{2a}$. The minimum value is $f\left(-\frac{b}{2a}\right)$.

2. If $a < 0$, f has a *maximum* at $x = -\frac{b}{2a}$. The maximum value is $f\left(-\frac{b}{2a}\right)$.

Example 5 The Maximum Height of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Algebraic Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c$$
$$= -0.0032x^2 + x + 3$$

which implies that $a = -0.0032$ and $b = 1$. Because $a < 0$, the function has a maximum when $x = -b/(2a)$. So, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$x = -\frac{b}{2a}$$
$$= -\frac{1}{2(-0.0032)}$$
$$= 156.25 \text{ feet.}$$

At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3$$
$$= 81.125 \text{ feet.}$$

CHECKPoint Now try Exercise 75.

Graphical Solution

Use a graphing utility to graph

$$y = -0.0032x^2 + x + 3$$

so that you can see the important features of the parabola. Use the *maximum* feature (see Figure 2.9) or the *zoom* and *trace* features (see Figure 2.10) of the graphing utility to approximate the maximum height on the graph to be $y \approx 81.125$ feet at $x \approx 156.25$.

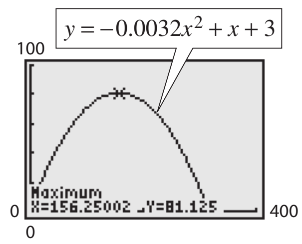


FIGURE 2.9

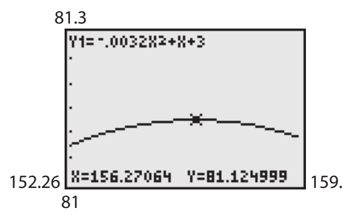


FIGURE 2.10

2.1 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. Linear, constant, and squaring functions are examples of ________ functions.

2. A polynomial function of degree n and leading coefficient a_n is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0 \ (a_n \neq 0)$ where n is a ________ ________ and $a_n, a_{n-1}, \ldots, a_1, a_0$ are ________ numbers.

3. A ________ function is a second-degree polynomial function, and its graph is called a ________.

4. The graph of a quadratic function is symmetric about its ________.

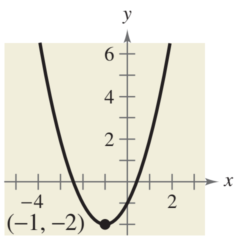
5. If the graph of a quadratic function opens upward, then its leading coefficient is ________ and the vertex of the graph is a ________.

6. If the graph of a quadratic function opens downward, then its leading coefficient is ________ and the vertex of the graph is a ________.

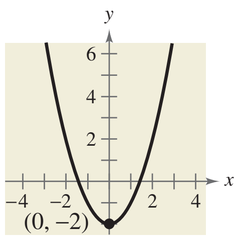
SKILLS AND APPLICATIONS

In Exercises 7–12, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)



(b)



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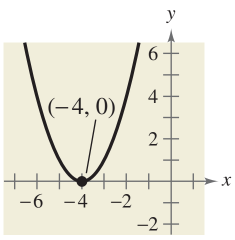
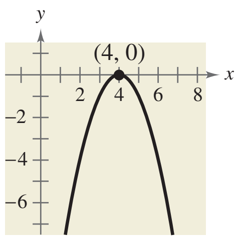


(d)



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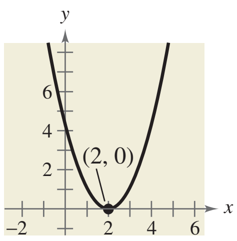

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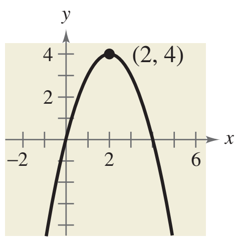


7. $f(x) = (x - 2)^2$

8. $f(x) = (x + 4)^2$

9. $f(x) = x^2 - 2$

10. $f(x) = (x + 1)^2 - 2$

11. $f(x) = 4 - (x - 2)^2$

12. $f(x) = -(x - 4)^2$

In Exercises 13–16, graph each function. Compare the graph of each function with the graph of $y = x^2$.

13. (a) $f(x) = \frac{1}{2}x^2$ (b) $g(x) = -\frac{1}{8}x^2$
(c) $h(x) = \frac{3}{2}x^2$ (d) $k(x) = -3x^2$

14. (a) $f(x) = x^2 + 1$ (b) $g(x) = x^2 - 1$
(c) $h(x) = x^2 + 3$ (d) $k(x) = x^2 - 3$

15. (a) $f(x) = (x - 1)^2$ (b) $g(x) = (3x)^2 + 1$
(c) $h(x) = \left(\frac{1}{3}x\right)^2 - 3$ (d) $k(x) = (x + 3)^2$

16. (a) $f(x) = -\frac{1}{2}(x - 2)^2 + 1$
(b) $g(x) = \left[\frac{1}{2}(x - 1)\right]^2 - 3$
(c) $h(x) = -\frac{1}{2}(x + 2)^2 - 1$
(d) $k(x) = [2(x + 1)]^2 + 4$

In Exercises 17–34, sketch the graph of the quadratic function without using a graphing utility. Identify the vertex, axis of symmetry, and x-intercept(s).

17. $f(x) = 1 - x^2$

18. $g(x) = x^2 - 8$

19. $f(x) = x^2 + 7$

20. $h(x) = 12 - x^2$

21. $f(x) = \frac{1}{2}x^2 - 4$

22. $f(x) = 16 - \frac{1}{4}x^2$

23. $f(x) = (x + 4)^2 - 3$

24. $f(x) = (x - 6)^2 + 8$

25. $h(x) = x^2 - 8x + 16$

26. $g(x) = x^2 + 2x + 1$

27. $f(x) = x^2 - x + \frac{5}{4}$

28. $f(x) = x^2 + 3x + \frac{1}{4}$

29. $f(x) = -x^2 + 2x + 5$

30. $f(x) = -x^2 - 4x + 1$

31. $h(x) = 4x^2 - 4x + 21$

32. $f(x) = 2x^2 - x + 1$

33. $f(x) = \frac{1}{4}x^2 - 2x - 12$

34. $f(x) = -\frac{1}{3}x^2 + 3x - 6$

In Exercises 35–42, use a graphing utility to graph the quadratic function. Identify the vertex, axis of symmetry, and x-intercepts. Then check your results algebraically by writing the quadratic function in standard form.

35. $f(x) = -(x^2 + 2x - 3)$

36. $f(x) = -(x^2 + x - 30)$

37. $g(x) = x^2 + 8x + 11$

38. $f(x) = x^2 + 10x + 14$

39. $f(x) = 2x^2 - 16x + 31$

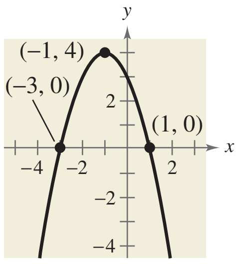
40. $f(x) = -4x^2 + 24x - 41$

41. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$

42. $f(x) = \frac{3}{5}(x^2 + 6x - 5)$

In Exercises 43–46, write an equation for the parabola in standard form.

43.



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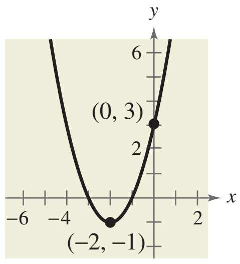

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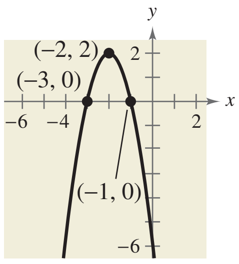

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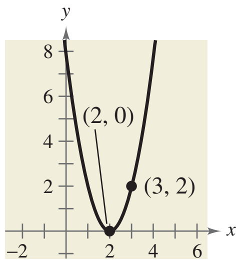


In Exercises 47–56, write the standard form of the equation of the parabola that has the indicated vertex and whose graph passes through the given point.

47. Vertex: $(-2, 5)$; point: $(0, 9)$

48. Vertex: $(4, -1)$; point: $(2, 3)$

49. Vertex: $(1, -2)$; point: $(-1, 14)$

50. Vertex: $(2, 3)$; point: $(0, 2)$

51. Vertex: $(5, 12)$; point: $(7, 15)$

52. Vertex: $(-2, -2)$; point: $(-1, 0)$

53. Vertex: $\left(-\frac{1}{4}, \frac{3}{2}\right)$; point: $(-2, 0)$

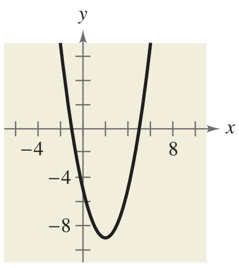
54. Vertex: $\left(\frac{5}{2}, -\frac{3}{4}\right)$; point: $(-2, 4)$

55. Vertex: $\left(-\frac{5}{2}, 0\right)$; point: $\left(-\frac{7}{2}, -\frac{16}{3}\right)$

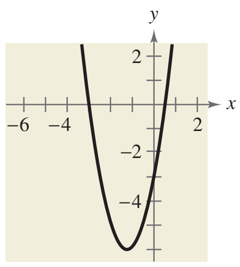
56. Vertex: $(6, 6)$; point: $\left(\frac{61}{10}, \frac{3}{2}\right)$

GRAPHICAL REASONING In Exercises 57 and 58, determine the x-intercept(s) of the graph visually. Then find the x-intercept(s) algebraically to confirm your results.

57. $y = x^2 - 4x - 5$



58. $y = 2x^2 + 5x - 3$



In Exercises 59–64, use a graphing utility to graph the quadratic function. Find the x-intercepts of the graph and compare them with the solutions of the corresponding quadratic equation when $f(x) = 0$.

59. $f(x) = x^2 - 4x$

60. $f(x) = -2x^2 + 10x$

61. $f(x) = x^2 - 9x + 18$

62. $f(x) = x^2 - 8x - 20$

63. $f(x) = 2x^2 - 7x - 30$

64. $f(x) = \frac{7}{10}(x^2 + 12x - 45)$

In Exercises 65–70, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given x-intercepts. (There are many correct answers.)

65. $(-1, 0), (3, 0)$

66. $(-5, 0), (5, 0)$

67. $(0, 0), (10, 0)$

68. $(4, 0), (8, 0)$

69. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

70. $\left(-\frac{5}{2}, 0\right), (2, 0)$

In Exercises 71–74, find two positive real numbers whose product is a maximum.

71. The sum is 110.

72. The sum is S.

73. The sum of the first and twice the second is 24.

74. The sum of the first and three times the second is 42.

75. PATH OF A DIVER The path of a diver is given by

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where y is the height (in feet) and x is the horizontal distance from the end of the diving board (in feet). What is the maximum height of the diver?

76. HEIGHT OF A BALL The height y (in feet) of a punted football is given by

$$y = -\frac{16}{2025}x^2 + \frac{9}{5}x + 1.5$$

where x is the horizontal distance (in feet) from the point at which the ball is punted.

(a) How high is the ball when it is punted?

(b) What is the maximum height of the punt?

(c) How long is the punt?

77. MINIMUM COST A manufacturer of lighting fixtures has daily production costs of $C = 800 - 10x + 0.25x^2$, where C is the total cost (in dollars) and x is the number of units produced. How many fixtures should be produced each day to yield a minimum cost?

78. MAXIMUM PROFIT The profit P (in hundreds of dollars) that a company makes depends on the amount x (in hundreds of dollars) the company spends on advertising according to the model $P = 230 + 20x - 0.5x^2$. What expenditure for advertising will yield a maximum profit?

79. MAXIMUM REVENUE The total revenue R earned (in thousands of dollars) from manufacturing handheld video games is given by

$$R(p) = -25p^2 + 1200p$$

where p is the price per unit (in dollars).

(a) Find the revenues when the price per unit is \$20, \$25, and \$30.

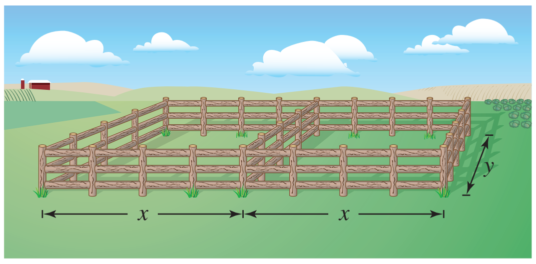
(b) Find the unit price that will yield a maximum revenue. What is the maximum revenue? Explain your results.

80. MAXIMUM REVENUE The total revenue R earned per day (in dollars) from a pet-sitting service is given by $R(p) = -12p^2 + 150p$, where p is the price charged per pet (in dollars).

(a) Find the revenues when the price per pet is \$4, \$6, and \$8.

(b) Find the price that will yield a maximum revenue. What is the maximum revenue? Explain your results.

81. NUMERICAL, GRAPHICAL, AND ANALYTICAL ANALYSIS A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure).



(a) Write the area A of the corrals as a function of x.

(b) Create a table showing possible values of x and the corresponding areas of the corral. Use the table to estimate the dimensions that will produce the maximum enclosed area.

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

(d) Write the area function in standard form to find analytically the dimensions that will produce the maximum area.

(e) Compare your results from parts (b), (c), and (d).

82. GEOMETRY An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter single-lane running track.

(a) Draw a diagram that illustrates the problem. Let x and y represent the length and width of the rectangular region, respectively.

(b) Determine the radius of each semicircular end of the room. Determine the distance, in terms of y, around the inside edge of each semicircular part of the track.

(c) Use the result of part (b) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.

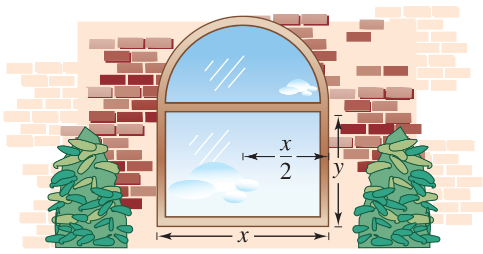
(d) Use the result of part (c) to write the area A of the rectangular region as a function of x. What dimensions will produce a rectangle of maximum area?

83. MAXIMUM REVENUE A small theater has a seating capacity of 2000. When the ticket price is \$20, attendance is 1500. For each \$1 decrease in price, attendance increases by 100.

(a) Write the revenue R of the theater as a function of ticket price x.

(b) What ticket price will yield a maximum revenue? What is the maximum revenue?

84. MAXIMUM AREA A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). The perimeter of the window is 16 feet.



(a) Write the area A of the window as a function of x.

(b) What dimensions will produce a window of maximum area?

85. GRAPHICAL ANALYSIS From 1950 through 2005, the per capita consumption C of cigarettes by Americans (age 18 and older) can be modeled by $C = 3565.0 + 60.30t - 1.783t^2$, $0 \le t \le 55$, where t is the year, with $t = 0$ corresponding to 1950. (Source: *Tobacco Outlook Report*)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 2005, the U.S. population (age 18 and over) was 296,329,000. Of those, about 59,858,458 were smokers. What was the average annual cigarette consumption *per smoker* in 2005? What was the average daily cigarette consumption *per smoker*?

86. DATA ANALYSIS: SALES The sales y (in billions of dollars) for Harley-Davidson from 2000 through 2007 are shown in the table. (Source: U.S. Harley-Davidson, Inc.)

Year	Sales, y
2000	2.91
2001	3.36
2002	4.09
2003	4.62
2004	5.02
2005	5.34
2006	5.80
2007	5.73

(a) Use a graphing utility to create a scatter plot of the data. Let x represent the year, with $x = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model in the same viewing window as the scatter plot. How well does the model fit the data?

(d) Use the *trace* feature of the graphing utility to approximate the year in which the sales for Harley-Davidson were the greatest.

(e) Verify your answer to part (d) algebraically.

(f) Use the model to predict the sales for Harley-Davidson in 2010.

EXPLORATION

TRUE OR FALSE? In Exercises 87–90, determine whether the statement is true or false. Justify your answer.

87. The function given by $f(x) = -12x^2 - 1$ has no x-intercepts.

88. The graphs of $f(x) = -4x^2 - 10x + 7$ and $g(x) = 12x^2 + 30x + 1$ have the same axis of symmetry.

89. The graph of a quadratic function with a negative leading coefficient will have a maximum value at its vertex.

90. The graph of a quadratic function with a positive leading coefficient will have a minimum value at its vertex.

THINK ABOUT IT In Exercises 91–94, find the values of b such that the function has the given maximum or minimum value.

91. $f(x) = -x^2 + bx - 75$; Maximum value: 25

92. $f(x) = -x^2 + bx - 16$; Maximum value: 48

93. $f(x) = x^2 + bx + 26$; Minimum value: 10

94. $f(x) = x^2 + bx - 25$; Minimum value: -50

95. Write the quadratic function

$$f(x) = ax^2 + bx + c$$

in standard form to verify that the vertex occurs at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

96. CAPSTONE The profit P (in millions of dollars) for a recreational vehicle retailer is modeled by a quadratic function of the form

$$P = at^2 + bt + c$$

where t represents the year. If you were president of the company, which of the models below would you prefer? Explain your reasoning.

(a) a is positive and $-b/(2a) \le t$.

(b) a is positive and $t \le -b/(2a)$.

(c) a is negative and $-b/(2a) \le t$.

(d) a is negative and $t \le -b/(2a)$.

97. GRAPHICAL ANALYSIS

(a) Graph $y = ax^2$ for $a = -2, -1, -0.5, 0.5, 1$ and 2. How does changing the value of a affect the graph?

(b) Graph $y = (x - h)^2$ for $h = -4, -2, 2$, and 4. How does changing the value of h affect the graph?

(c) Graph $y = x^2 + k$ for $k = -4, -2, 2$, and 4. How does changing the value of k affect the graph?

98. Describe the sequence of transformation from f to g given that $f(x) = x^2$ and $g(x) = a(x - h)^2 + k$. (Assume a, h, and k are positive.)

99. Is it possible for a quadratic equation to have only one x-intercept? Explain.

100. Assume that the function given by

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

has two real zeros. Show that the x-coordinate of the vertex of the graph is the average of the zeros of f. (*Hint:* Use the Quadratic Formula.)

PROJECT: HEIGHT OF A BASKETBALL To work an extended application analyzing the height of a basketball after it has been dropped, visit this text's website at *academic.cengage.com.*

2.2 Polynomial Functions of Higher Degree

What you should learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

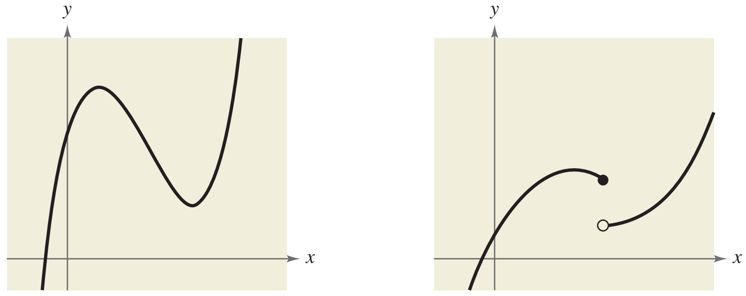
Why you should learn it

You can use polynomial functions to analyze business situations such as how revenue is related to advertising expenses, as discussed in Exercise 104 on page 148.

Bill Aron/PhotoEdit, Inc.

Graphs of Polynomial Functions

In this section, you will study basic features of the graphs of polynomial functions. The first feature is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.11(a). The graph shown in Figure 2.11(b) is an example of a piecewise-defined function that is not continuous.

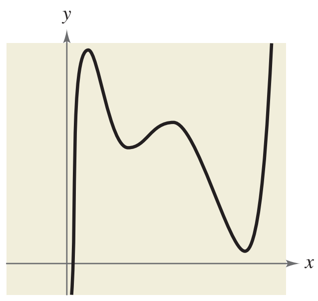


(a) Polynomial functions have continuous graphs.

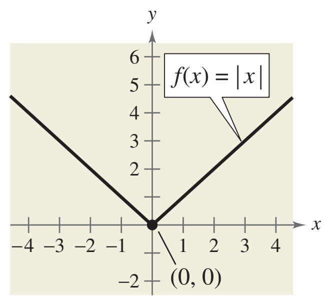
(b) Functions with graphs that are not continuous are not polynomial functions.

FIGURE 2.11

The second feature is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 2.12. A polynomial function cannot have a sharp turn. For instance, the function given by $f(x) = |x|$, which has a sharp turn at the point $(0, 0)$, as shown in Figure 2.13, is not a polynomial function.



Polynomial functions have graphs with smooth, rounded turns.
FIGURE 2.12

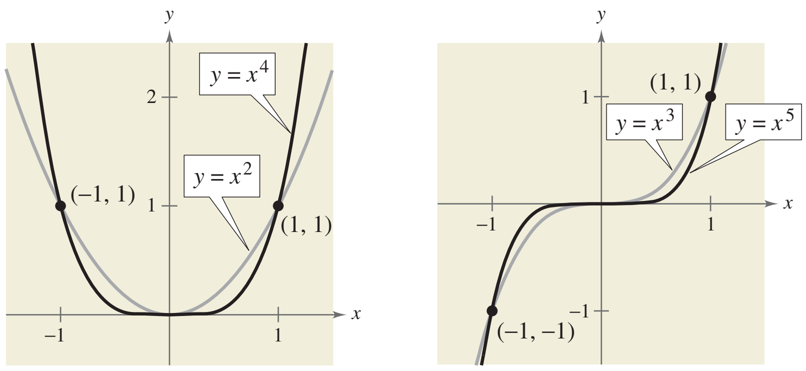


Graphs of polynomial functions cannot have sharp turns.
FIGURE 2.13

The graphs of polynomial functions of degree greater than 2 are more difficult to analyze than the graphs of polynomials of degree 0, 1, or 2. However, using the features presented in this section, coupled with your knowledge of point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

> **Study Tip**
>
> For power functions given by $f(x) = x^n$, if n is even, then the graph of the function is symmetric with respect to the y-axis, and if n is odd, then the graph of the function is symmetric with respect to the origin.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. From Figure 2.14, you can see that when n is *even*, the graph is similar to the graph of $f(x) = x^2$, and when n is *odd*, the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph near the origin. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**



(a) If n is even, the graph of $y = x^n$ touches the axis at the x-intercept.

(b) If n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

FIGURE 2.14

Example 1 Sketching Transformations of Polynomial Functions

Sketch the graph of each function.

a. $f(x) = -x^5$ **b.** $h(x) = (x + 1)^4$

Solution

a. Because the degree of $f(x) = -x^5$ is odd, its graph is similar to the graph of $y = x^3$. In Figure 2.15, note that the negative coefficient has the effect of reflecting the graph in the x-axis.

b. The graph of $h(x) = (x + 1)^4$, as shown in Figure 2.16, is a left shift by one unit of the graph of $y = x^4$.

> **Algebra Help**
>
> You can review the techniques for shifting, reflecting, and stretching graphs in Section 1.7.

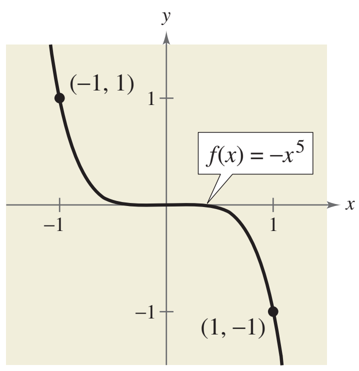


FIGURE 2.15

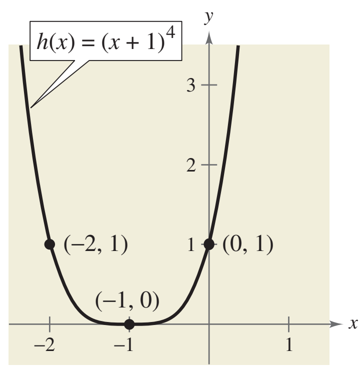


FIGURE 2.16

CHECKPoint Now try Exercise 17.

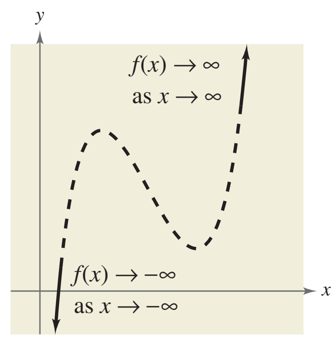
The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial function eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

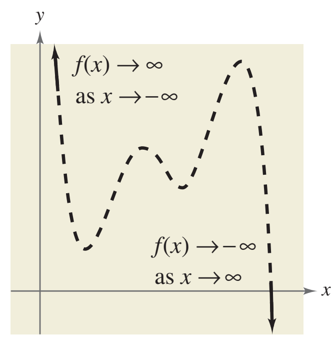
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$ eventually rises or falls in the following manner.

1. When n is *odd:*

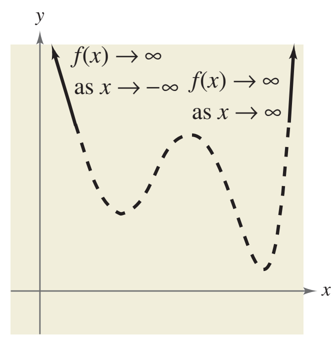


If the leading coefficient is positive ($a_n > 0$), the graph falls to the left and rises to the right.

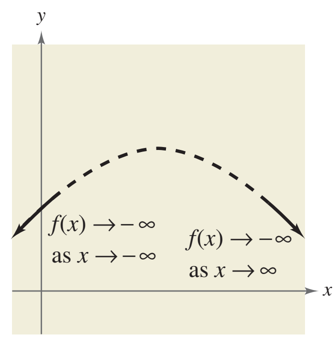


If the leading coefficient is negative ($a_n < 0$), the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive ($a_n > 0$), the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), the graph falls to the left and right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

Study Tip

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

WARNING / CAUTION

A polynomial function is written in **standard form** if its terms are written in descending order of exponents from left to right. Before applying the Leading Coefficient Test to a polynomial function, it is a good idea to make sure that the polynomial function is written in standard form.

Example 2 Applying the Leading Coefficient Test

Describe the right-hand and left-hand behavior of the graph of each function.

a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

a. Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 2.17.

b. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 2.18.

c. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 2.19.

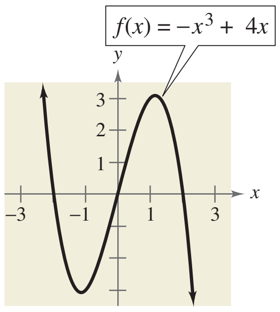


FIGURE 2.17

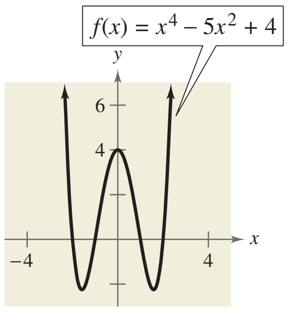


FIGURE 2.18

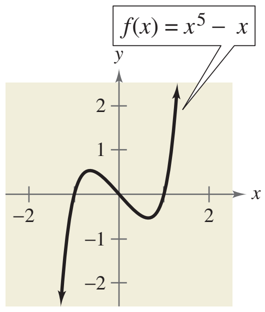


FIGURE 2.19

CHECKPoint Now try Exercise 23.

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

Study Tip

Remember that the *zeros* of a function of x are the x-values for which the function is zero.

1. The function f has, at most, n real zeros. (You will study this result in detail in the discussion of the Fundamental Theorem of Algebra in Section 2.5.)
2. The graph of f has, at most, $n - 1$ turning points. (Turning points, also called relative minima or relative maxima, are points at which the graph changes from increasing to decreasing or vice versa.)

Finding the zeros of polynomial functions is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros, and in other cases you can use information about the zeros of a function to help sketch its graph. Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts.

Algebra Help

To do Example 3 algebraically, you need to be able to completely factor polynomials. You can review the techniques for factoring in Appendix A.3.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an *x-intercept* of the graph of f.

Example 3 Finding the Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = -2x^4 + 2x^2.$$

Then determine the number of turning points of the graph of the function.

Algebraic Solution

To find the real zeros of the function, set $f(x)$ equal to zero and solve for x.

$$-2x^4 + 2x^2 = 0 \quad \text{Set } f(x) \text{ equal to 0.}$$

$$-2x^2(x^2 - 1) = 0 \quad \text{Remove common monomial factor.}$$

$$-2x^2(x - 1)(x + 1) = 0 \quad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$. Because the function is a fourth-degree polynomial, the graph of f can have at most $4 - 1 = 3$ turning points.

Graphical Solution

Use a graphing utility to graph $y = -2x^4 + 2x^2$. In Figure 2.20, the graph appears to have zeros at $(0, 0)$, $(1, 0)$, and $(-1, 0)$. Use the *zero* or *root* feature, or the *zoom* and *trace* features, of the graphing utility to verify these zeros. So, the real zeros are $x = 0$, $x = 1$, and $x = -1$. From the figure, you can see that the graph has three turning points. This is consistent with the fact that a fourth-degree polynomial can have at most three turning points.

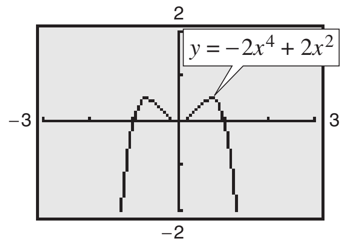


FIGURE 2.20

CHECKPoint Now try Exercise 35.

In Example 3, note that because the exponent is greater than 1, the factor $-2x^2$ yields the *repeated* zero $x = 0$. Because the exponent is even, the graph touches the x-axis at $x = 0$, as shown in Figure 2.20.

Repeated Zeros

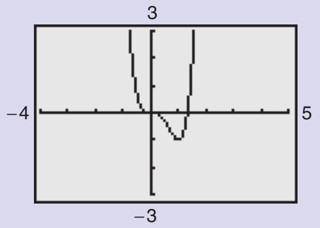
A factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, the graph *crosses* the x-axis at $x = a$.
2. If k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

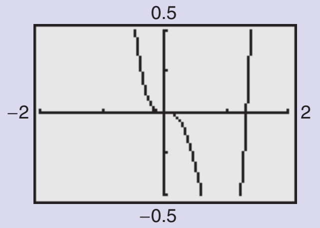
TECHNOLOGY

Example 4 uses an *algebraic approach* to describe the graph of the function. A graphing utility is a complement to this approach. Remember that an important aspect of using a graphing utility is to find a viewing window that shows all significant features of the graph. For instance, the viewing window in part (a) illustrates all of the significant features of the function in Example 4 while the viewing window in part (b) does not.

a.



b.



To graph polynomial functions, you can use the fact that a polynomial function can change signs only at its zeros. Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. (This follows from the Intermediate Value Theorem, which you will study later in this section.) This means that when the real zeros of a polynomial function are put in order, they divide the real number line into intervals in which the function has no sign changes. These resulting intervals are **test intervals** in which a representative x-value in the interval is chosen to determine if the value of the polynomial function is positive (the graph lies above the x-axis) or negative (the graph lies below the x-axis).

Example 4 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.21).
2. *Find the Zeros of the Polynomial.* By factoring $f(x) = 3x^4 - 4x^3$ as $f(x) = x^3(3x - 4)$, you can see that the zeros of f are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.21.
3. *Plot a Few Additional Points.* Use the zeros of the polynomial to find the test intervals. In each test interval, choose a representative x-value and evaluate the polynomial function, as shown in the table.

Test interval	Representative x-value	Value of f	Sign	Point on graph
$(-\infty, 0)$	-1	$f(-1) = 7$	Positive	$(-1, 7)$
$\left(0, \frac{4}{3}\right)$	1	$f(1) = -1$	Negative	$(1, -1)$
$\left(\frac{4}{3}, \infty\right)$	1.5	$f(1.5) = 1.6875$	Positive	$(1.5, 1.6875)$

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.22. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$.

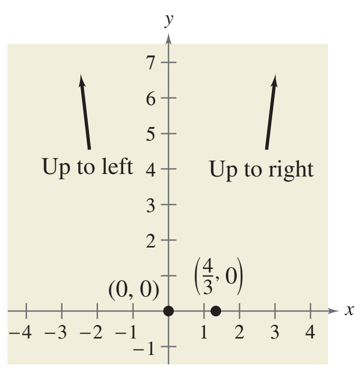


FIGURE 2.21

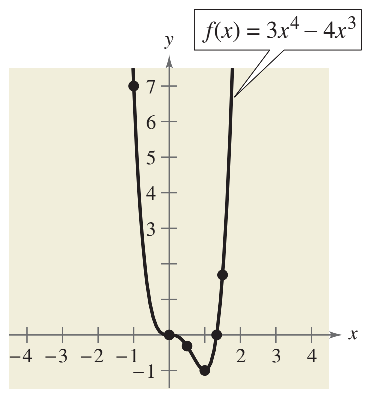


FIGURE 2.22

CHECK*Point* Now try Exercise 75.

WARNING / CAUTION

If you are unsure of the shape of a portion of the graph of a polynomial function, plot some additional points, such as the point $(0.5, -0.3125)$, as shown in Figure 2.22.

Example 5 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 2.23).

2. *Find the Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

you can see that the zeros of f are $x = 0$ (odd multiplicity) and $x = \frac{3}{2}$ (even multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.23.

3. *Plot a Few Additional Points.* Use the zeros of the polynomial to find the test intervals. In each test interval, choose a representative x-value and evaluate the polynomial function, as shown in the table.

Test interval	Representative x-value	Value of f	Sign	Point on graph
$(-\infty, 0)$	-0.5	$f(-0.5) = 4$	Positive	$(-0.5, 4)$
$\left(0, \frac{3}{2}\right)$	0.5	$f(0.5) = -1$	Negative	$(0.5, -1)$
$\left(\frac{3}{2}, \infty\right)$	2	$f(2) = -1$	Negative	$(2, -1)$

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.24. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

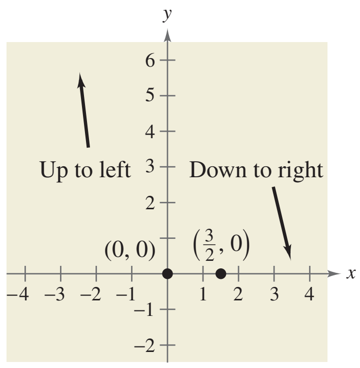


FIGURE 2.23

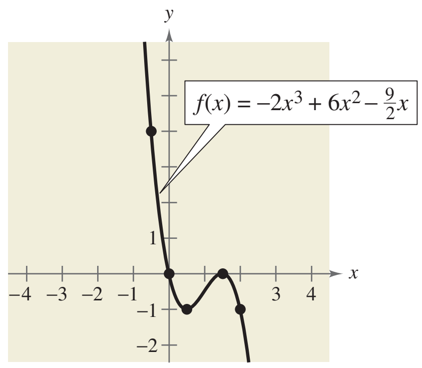


FIGURE 2.24

CHECKPoint Now try Exercise 77.

Study Tip

Observe in Example 5 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if the zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side of the zero to the other side. If the zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side.

The Intermediate Value Theorem

The next theorem, called the **Intermediate Value Theorem,** illustrates the existence of real zeros of polynomial functions. This theorem implies that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 2.25.)

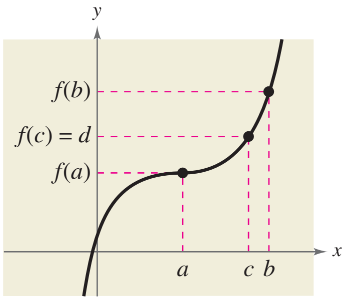


FIGURE 2.25

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function given by $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown in Figure 2.26.

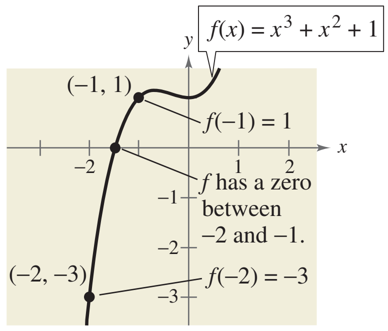


FIGURE 2.26

By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired accuracy. This concept is further demonstrated in Example 6.

Example 6 Approximating a Zero of a Polynomial Function

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - x^2 + 1.$$

Solution

Begin by computing a few function values, as follows.

x	$f(x)$
-2	-11
-1	-1
0	1
1	1

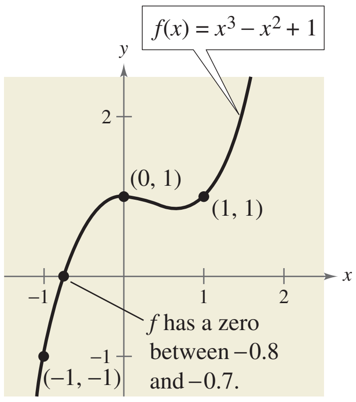


FIGURE 2.27

Because $f(-1)$ is negative and $f(0)$ is positive, you can apply the Intermediate Value Theorem to conclude that the function has a zero between -1 and 0. To pinpoint this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each point. When you do this, you will find that

$$f(-0.8) = -0.152 \quad \text{and} \quad f(-0.7) = 0.167.$$

So, f must have a zero between -0.8 and -0.7, as shown in Figure 2.27. For a more accurate approximation, compute function values between $f(-0.8)$ and $f(-0.7)$ and apply the Intermediate Value Theorem again. By continuing this process, you can approximate this zero to any desired accuracy.

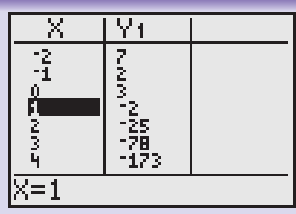
CHECKPoint Now try Exercise 93.

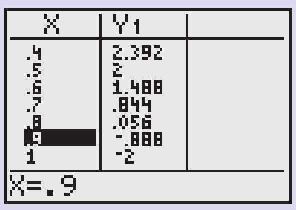
TECHNOLOGY

You can use the *table* feature of a graphing utility to approximate the zeros of a polynomial function. For instance, for the function given by

$$f(x) = -2x^3 - 3x^2 + 3$$

create a table that shows the function values for $-20 \le x \le 20$, as shown in the first table at the right. Scroll through the table looking for consecutive function values that differ in sign. From the table, you can see that $f(0)$ and $f(1)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between 0 and 1. You can adjust your table to show function values for $0 \le x \le 1$ using increments of 0.1, as shown in the second table at the right. By scrolling through the table you can see that $f(0.8)$ and $f(0.9)$ differ in sign. So, the function has a zero between 0.8 and 0.9. If you repeat this process several times, you should obtain $x \approx 0.806$ as the zero of the function. Use the *zero* or *root* feature of a graphing utility to confirm this result.





2.2 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. The graphs of all polynomial functions are ________, which means that the graphs have no breaks, holes, or gaps.

2. The ________ ________ ________ is used to determine the left-hand and right-hand behavior of the graph of a polynomial function.

3. Polynomial functions of the form $f(x) =$ ________ are often referred to as power functions.

4. A polynomial function of degree n has at most ________ real zeros and at most ________ turning points.

5. If $x = a$ is a zero of a polynomial function f, then the following three statements are true.

(a) $x = a$ is a ________ of the polynomial equation $f(x) = 0$.

(b) ________ is a factor of the polynomial $f(x)$.

(c) $(a, 0)$ is an ________ of the graph of f.

6. If a real zero of a polynomial function is of even multiplicity, then the graph of f ________ the x-axis at $x = a$, and if it is of odd multiplicity, then the graph of f ________ the x-axis at $x = a$.

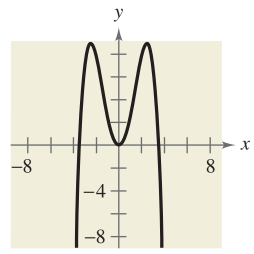
7. A polynomial function is written in ________ form if its terms are written in descending order of exponents from left to right.

8. The ________ ________ Theorem states that if f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

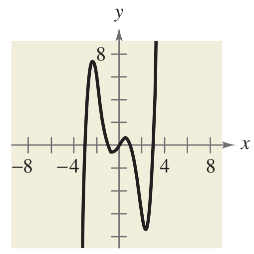
SKILLS AND APPLICATIONS

In Exercises 9–16, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

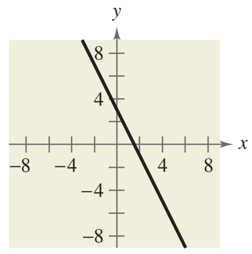
(a)



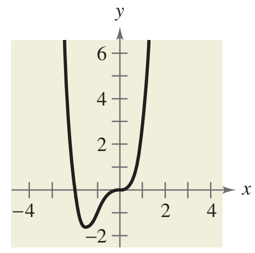
(b)



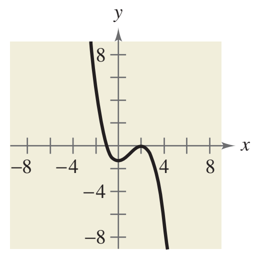
(c)



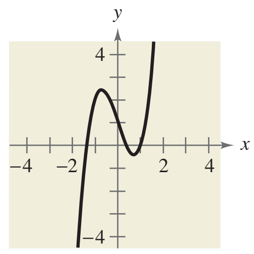
(d)



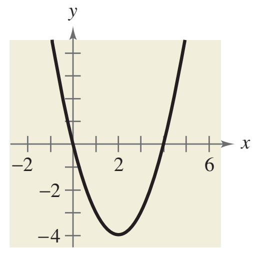
(e)



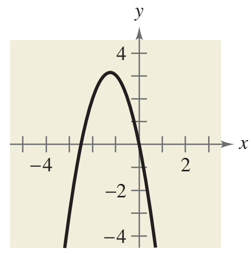
(f)



(g)



(h)



9. $f(x) = -2x + 3$

10. $f(x) = x^2 - 4x$

11. $f(x) = -2x^2 - 5x$

12. $f(x) = 2x^3 - 3x + 1$

13. $f(x) = -\frac{1}{4}x^4 + 3x^2$

14. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$

15. $f(x) = x^4 + 2x^3$

16. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

In Exercises 17–20, sketch the graph of $y = x^n$ and each transformation.

17. $y = x^3$

(a) $f(x) = (x - 4)^3$ (b) $f(x) = x^3 - 4$

(c) $f(x) = -\frac{1}{4}x^3$ (d) $f(x) = (x - 4)^3 - 4$

18. $y = x^5$

(a) $f(x) = (x + 1)^5$ (b) $f(x) = x^5 + 1$

(c) $f(x) = 1 - \frac{1}{2}x^5$ (d) $f(x) = -\frac{1}{2}(x + 1)^5$

19. $y = x^4$

(a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 3$

(c) $f(x) = 4 - x^4$ (d) $f(x) = \frac{1}{2}(x - 1)^4$

(e) $f(x) = (2x)^4 + 1$ (f) $f(x) = \left(\frac{1}{2}x\right)^4 - 2$

20. $y = x^6$

(a) $f(x) = -\frac{1}{8}x^6$ (b) $f(x) = (x + 2)^6 - 4$

(c) $f(x) = x^6 - 5$ (d) $f(x) = -\frac{1}{4}x^6 + 1$

(e) $f(x) = \left(\frac{1}{4}x\right)^6 - 2$ (f) $f(x) = (2x)^6 - 1$

In Exercises 21–30, describe the right-hand and left-hand behavior of the graph of the polynomial function.

21. $f(x) = \frac{1}{5}x^3 + 4x$ **22.** $f(x) = 2x^2 - 3x + 1$

23. $g(x) = 5 - \frac{7}{2}x - 3x^2$ **24.** $h(x) = 1 - x^6$

25. $f(x) = -2.1x^5 + 4x^3 - 2$

26. $f(x) = 4x^5 - 7x + 6.5$

27. $f(x) = 6 - 2x + 4x^2 - 5x^3$

28. $f(x) = (3x^4 - 2x + 5)/4$

29. $h(t) = -\frac{3}{4}(t^2 - 3t + 6)$

30. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

GRAPHICAL ANALYSIS In Exercises 31–34, use a graphing utility to graph the functions *f* and *g* in the same viewing window. Zoom out sufficiently far to show that the right-hand and left-hand behaviors of *f* and *g* appear identical.

31. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

32. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

33. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

34. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

In Exercises 35–50, (a) find all the real zeros of the polynomial function, (b) determine the multiplicity of each zero and the number of turning points of the graph of the function, and (c) use a graphing utility to graph the function and verify your answers.

35. $f(x) = x^2 - 36$ **36.** $f(x) = 81 - x^2$

37. $h(t) = t^2 - 6t + 9$ **38.** $f(x) = x^2 + 10x + 25$

39. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$ **40.** $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

41. $f(x) = 3x^3 - 12x^2 + 3x$ **42.** $g(x) = 5x(x^2 - 2x - 1)$

43. $f(t) = t^3 - 8t^2 + 16t$ **44.** $f(x) = x^4 - x^3 - 30x^2$

45. $g(t) = t^5 - 6t^3 + 9t$ **46.** $f(x) = x^5 + x^3 - 6x$

47. $f(x) = 3x^4 + 9x^2 + 6$ **48.** $f(x) = 2x^4 - 2x^2 - 40$

49. $g(x) = x^3 + 3x^2 - 4x - 12$

50. $f(x) = x^3 - 4x^2 - 25x + 100$

GRAPHICAL ANALYSIS In Exercises 51–54, (a) use a graphing utility to graph the function, (b) use the graph to approximate any *x*-intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the results of part (c) with any *x*-intercepts of the graph.

51. $y = 4x^3 - 20x^2 + 25x$

52. $y = 4x^3 + 4x^2 - 8x - 8$

53. $y = x^5 - 5x^3 + 4x$ **54.** $y = \frac{1}{4}x^3(x^2 - 9)$

In Exercises 55–64, find a polynomial function that has the given zeros. (There are many correct answers.)

55. 0, 8 **56.** 0, -7

57. 2, -6 **58.** -4, 5

59. 0, -4, -5 **60.** 0, 1, 10

61. 4, -3, 3, 0 **62.** -2, -1, 0, 1, 2

63. $1 + \sqrt{3}, 1 - \sqrt{3}$ **64.** $2, 4 + \sqrt{5}, 4 - \sqrt{5}$

In Exercises 65–74, find a polynomial of degree *n* that has the given zero(s). (There are many correct answers.)

	Zero(s)	*Degree*
65.	$x = -3$	$n = 2$
66.	$x = -12, -6$	$n = 2$
67.	$x = -5, 0, 1$	$n = 3$
68.	$x = -2, 4, 7$	$n = 3$
69.	$x = 0, \sqrt{3}, -\sqrt{3}$	$n = 3$
70.	$x = 9$	$n = 3$
71.	$x = -5, 1, 2$	$n = 4$
72.	$x = -4, -1, 3, 6$	$n = 4$
73.	$x = 0, -4$	$n = 5$
74.	$x = -1, 4, 7, 8$	$n = 5$

In Exercises 75–88, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

75. $f(x) = x^3 - 25x$ **76.** $g(x) = x^4 - 9x^2$

77. $f(t) = \frac{1}{4}(t^2 - 2t + 15)$

78. $g(x) = -x^2 + 10x - 16$

79. $f(x) = x^3 - 2x^2$ **80.** $f(x) = 8 - x^3$

81. $f(x) = 3x^3 - 15x^2 + 18x$

82. $f(x) = -4x^3 + 4x^2 + 15x$

83. $f(x) = -5x^2 - x^3$ **84.** $f(x) = -48x^2 + 3x^4$

85. $f(x) = x^2(x - 4)$ **86.** $h(x) = \frac{1}{3}x^3(x - 4)^2$

87. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

88. $g(x) = \frac{1}{10}(x + 1)^2(x - 3)^3$

In Exercises 89–92, use a graphing utility to graph the function. Use the *zero* or *root* feature to approximate the real zeros of the function. Then determine the multiplicity of each zero.

89. $f(x) = x^3 - 16x$ **90.** $f(x) = \frac{1}{4}x^4 - 2x^2$

91. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

92. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

In Exercises 93–96, use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of the graphing utility to verify your results.

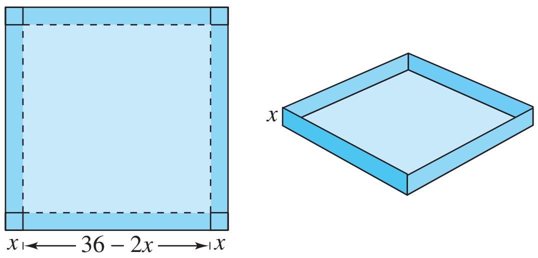
93. $f(x) = x^3 - 3x^2 + 3$

94. $f(x) = 0.11x^3 - 2.07x^2 + 9.81x - 6.88$

95. $g(x) = 3x^4 + 4x^3 - 3$

96. $h(x) = x^4 - 10x^2 + 3$

97. NUMERICAL AND GRAPHICAL ANALYSIS An open box is to be made from a square piece of material, 36 inches on a side, by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).



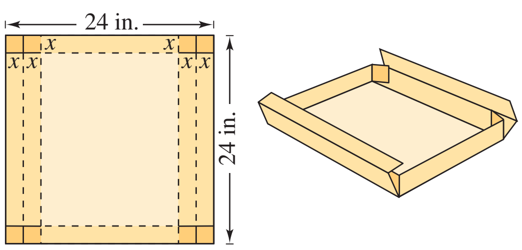
(a) Write a function $V(x)$ that represents the volume of the box.

(b) Determine the domain of the function.

(c) Use a graphing utility to create a table that shows box heights x and the corresponding volumes V. Use the table to estimate the dimensions that will produce a maximum volume.

(d) Use a graphing utility to graph V and use the graph to estimate the value of x for which $V(x)$ is maximum. Compare your result with that of part (c).

98. MAXIMUM VOLUME An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is to be done by cutting equal squares from the corners and folding along the dashed lines shown in the figure.

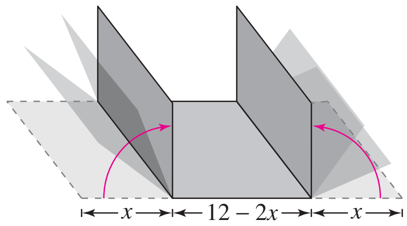


(a) Write a function $V(x)$ that represents the volume of the box.

(b) Determine the domain of the function V.

(c) Sketch a graph of the function and estimate the value of x for which $V(x)$ is maximum.

99. CONSTRUCTION A roofing contractor is fabricating gutters from 12-inch aluminum sheeting. The contractor plans to use an aluminum siding folding press to create the gutter by creasing equal lengths for the sidewalls (see figure).



(a) Let x represent the height of the sidewall of the gutter. Write a function A that represents the cross-sectional area of the gutter.

(b) The length of the aluminum sheeting is 16 feet. Write a function V that represents the volume of one run of gutter in terms of x.

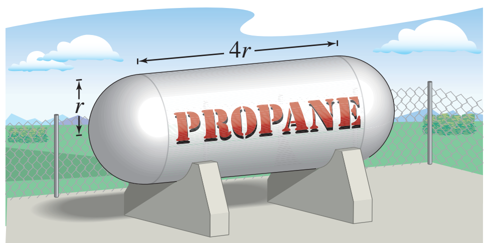
(c) Determine the domain of the function in part (b).

(d) Use a graphing utility to create a table that shows sidewall heights x and the corresponding volumes V. Use the table to estimate the dimensions that will produce a maximum volume.

(e) Use a graphing utility to graph V. Use the graph to estimate the value of x for which $V(x)$ is a maximum. Compare your result with that of part (d).

(f) Would the value of x change if the aluminum sheeting were of different lengths? Explain.

100. CONSTRUCTION An industrial propane tank is formed by adjoining two hemispheres to the ends of a right circular cylinder. The length of the cylindrical portion of the tank is four times the radius of the hemispherical components (see figure).



(a) Write a function that represents the total volume V of the tank in terms of r.

(b) Find the domain of the function.

(c) Use a graphing utility to graph the function.

(d) The total volume of the tank is to be 120 cubic feet. Use the graph from part (c) to estimate the radius and length of the cylindrical portion of the tank.

101. REVENUE The total revenues R (in millions of dollars) for Krispy Kreme from 2000 through 2007 are shown in the table.

Year	Revenue, R
2000	300.7
2001	394.4
2002	491.5
2003	665.6
2004	707.8
2005	543.4
2006	461.2
2007	429.3

A model that represents these data is given by $R = 3.0711t^4 - 42.803t^3 + 160.59t^2 - 62.6t + 307$, $0 \le t \le 7$, where t represents the year, with $t = 0$ corresponding to 2000. (Source: Krispy Kreme)

(a) Use a graphing utility to create a scatter plot of the data. Then graph the model in the same viewing window.

(b) How well does the model fit the data?

(c) Use a graphing utility to approximate any relative extrema of the model over its domain.

(d) Use a graphing utility to approximate the intervals over which the revenue for Krispy Kreme was increasing and decreasing over its domain.

(e) Use the results of parts (c) and (d) to write a short paragraph about Krispy Kreme's revenue during this time period.

102. REVENUE The total revenues R (in millions of dollars) for Papa John's International from 2000 through 2007 are shown in the table.

Year	Revenue, R
2000	944.7
2001	971.2
2002	946.2
2003	917.4
2004	942.4
2005	968.8
2006	1001.6
2007	1063.6

A model that represents these data is given by $R = -0.5635t^4 + 9.019t^3 - 40.20t^2 + 49.0t + 947$, $0 \le t \le 7$, where t represents the year, with $t = 0$ corresponding to 2000. (Source: Papa John's International)

(a) Use a graphing utility to create a scatter plot of the data. Then graph the model in the same viewing window.

(b) How well does the model fit the data?

(c) Use a graphing utility to approximate any relative extrema of the model over its domain.

(d) Use a graphing utility to approximate the intervals over which the revenue for Papa John's International was increasing and decreasing over its domain.

(e) Use the results of parts (c) and (d) to write a short paragraph about the revenue for Papa John's International during this time period.

103. TREE GROWTH The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years).

(a) Use a graphing utility to graph the function. (*Hint:* Use a viewing window in which $-10 \le x \le 45$ and $-5 \le y \le 60$.)

(b) Estimate the age of the tree when it is growing most rapidly. This point is called the *point of diminishing returns* because the increase in size will be less with each additional year.

(c) Using calculus, the point of diminishing returns can also be found by finding the vertex of the parabola given by

$$y = -0.009t^2 + 0.274t + 0.458.$$

Find the vertex of this parabola.

(d) Compare your results from parts (b) and (c).

104. REVENUE The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = \frac{1}{100,000}(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of this function, shown in the figure on the next page, to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expense above this amount will yield less return per dollar invested in advertising.

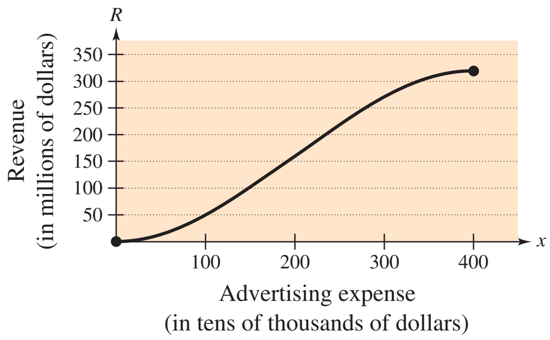


FIGURE FOR 104

EXPLORATION

TRUE OR FALSE? In Exercises 105–107, determine whether the statement is true or false. Justify your answer.

105. A fifth-degree polynomial can have five turning points in its graph.

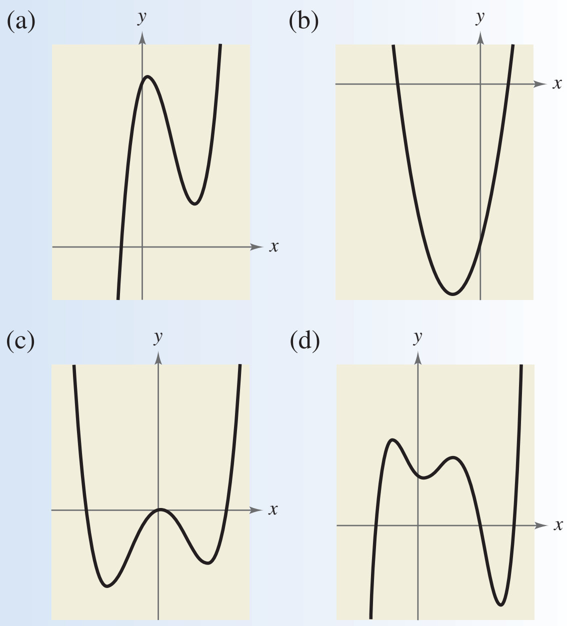
106. It is possible for a sixth-degree polynomial to have only one solution.

107. The graph of the function given by

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

108. CAPSTONE For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)



109. GRAPHICAL REASONING Sketch a graph of the function given by $f(x) = x^4$. Explain how the graph of each function g differs (if it does) from the graph of each function f. Determine whether g is odd, even, or neither.

(a) $g(x) = f(x) + 2$ (b) $g(x) = f(x + 2)$

(c) $g(x) = f(-x)$ (d) $g(x) = -f(x)$

(e) $g(x) = f\left(\frac{1}{2}x\right)$ (f) $g(x) = \frac{1}{2}f(x)$

(g) $g(x) = f\left(x^{3/4}\right)$ (h) $g(x) = (f \circ f)(x)$

110. THINK ABOUT IT For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree of the function and the sign of the leading coefficient of the function and the right-hand and left-hand behavior of the graph of the function.

(a) $f(x) = x^3 - 2x^2 - x + 1$

(b) $f(x) = 2x^5 + 2x^2 - 5x + 1$

(c) $f(x) = -2x^5 - x^2 + 5x + 3$

(d) $f(x) = -x^3 + 5x - 2$

(e) $f(x) = 2x^2 + 3x - 4$

(f) $f(x) = x^4 - 3x^2 + 2x - 1$

(g) $f(x) = x^2 + 3x + 2$

111. THINK ABOUT IT Sketch the graph of each polynomial function. Then count the number of zeros of the function and the numbers of relative minima and relative maxima. Compare these numbers with the degree of the polynomial. What do you observe?

(a) $f(x) = -x^3 + 9x$ (b) $f(x) = x^4 - 10x^2 + 9$

(c) $f(x) = x^5 - 16x$

112. Explore the transformations of the form

$$g(x) = a(x - h)^5 + k.$$

(a) Use a graphing utility to graph the functions $y_1 = -\frac{1}{3}(x - 2)^5 + 1$ and $y_2 = \frac{3}{5}(x + 2)^5 - 3$. Determine whether the graphs are increasing or decreasing. Explain.

(b) Will the graph of g always be increasing or decreasing? If so, is this behavior determined by a, h, or k? Explain.

(c) Use a graphing utility to graph the function given by $H(x) = x^5 - 3x^3 + 2x + 1$. Use the graph and the result of part (b) to determine whether H can be written in the form $H(x) = a(x - h)^5 + k$. Explain.

2.3 Polynomial and Synthetic Division

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder Theorem and the Factor Theorem.

Why you should learn it

Synthetic division can help you evaluate polynomial functions. For instance, in Exercise 85 on page 157, you will use synthetic division to determine the amount donated to support higher education in the United States in 2010.

MBI/Alamy

Long Division of Polynomials

In this section, you will study two procedures for *dividing* polynomials. These procedures are especially valuable in factoring and finding the zeros of polynomial functions. To begin, suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown in Figure 2.28. Because $x = 2$ is a zero of f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.

Example 1 Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\frac{6x^3}{x} = 6x^2$.

Think $\frac{-7x^2}{x} = -7x$.

Think $\frac{2x}{x} = 2$.

$$\begin{array}{r l}
 & 6x^2 - 7x + 2 \\
x - 2 \,\big)\!\! & 6x^3 - 19x^2 + 16x - 4 \\
 & \underline{6x^3 - 12x^2} \quad \text{Multiply: } 6x^2(x - 2). \\
 & -7x^2 + 16x \quad \text{Subtract.} \\
 & \underline{-7x^2 + 14x} \quad \text{Multiply: } -7x(x - 2). \\
 & 2x - 4 \quad \text{Subtract.} \\
 & \underline{2x - 4} \quad \text{Multiply: } 2(x - 2). \\
 & 0 \quad \text{Subtract.}
\end{array}$$

From this division, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

Note that this factorization agrees with the graph shown in Figure 2.28 in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

CHECKPoint Now try Exercise 11.

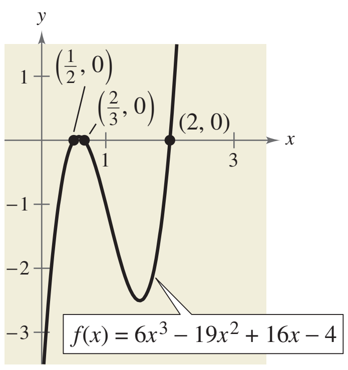


FIGURE 2.28

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$
\begin{array}{rrl}
 & x + 2 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow \quad x + 1 \,\big) & \overline{x^2 + 3x + 5} & \leftarrow \text{Dividend} \\
 & \underline{x^2 + x \phantom{{}+5}} & \\
 & 2x + 5 & \\
 & \underline{2x + 2} & \\
 & 3 & \leftarrow \text{Remainder}
\end{array}
$$

In fractional form, you can write this result as follows.

$$\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{\downarrow}\; 3}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\,\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Example 2 **Long Division of Polynomials**

Divide $x^3 - 1$ by $x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{r} x^2 + x + 1 \\ x - 1 \,\big)\overline{x^3 + 0x^2 + 0x - 1} \\ \underline{x^3 - x^2} \\ x^2 + 0x \\ \underline{x^2 - x} \\ x - 1 \\ \underline{x - 1} \\ 0 \end{array}$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \quad x \neq 1.$$

CHECKPoint Now try Exercise 17.

You can check the result of Example 2 by multiplying.

$$(x - 1)(x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1$$

Algebra Help

You can check a long division problem by multiplying. You can review the techniques for multiplying polynomials in Appendix A.3.

Example 3 **Long Division of Polynomials**

Divide $-5x^2 - 2 + 3x + 2x^4 + 4x^3$ by $2x - 3 + x^2$.

Solution

Begin by writing the dividend and divisor in descending powers of x.

$$\begin{array}{r} 2x^2 + 1 \\ x^2 + 2x - 3 \,\big)\overline{2x^4 + 4x^3 - 5x^2 + 3x - 2} \\ \underline{2x^4 + 4x^3 - 6x^2} \\ x^2 + 3x - 2 \\ \underline{x^2 + 2x - 3} \\ x + 1 \end{array}$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

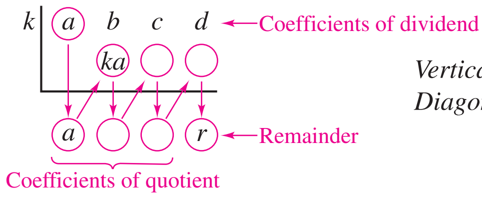
CHECKPoint Now try Exercise 23.

Synthetic Division

There is a nice shortcut for long division of polynomials by divisors of the form $x - k$. This shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

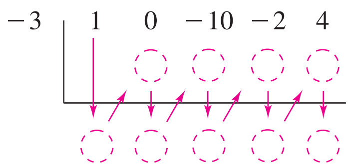
This algorithm for synthetic division works only for divisors of the form $x - k$. Remember that $x + k = x - (-k)$.

Example 4 Using Synthetic Division

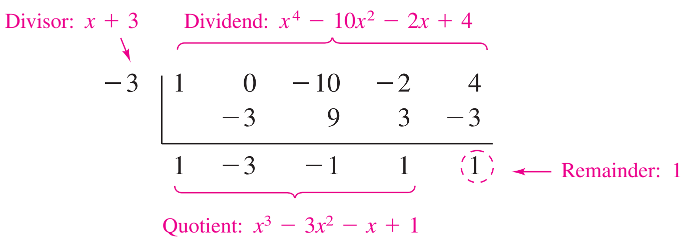
Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

CHECKPoint Now try Exercise 27.

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

For a proof of the Remainder Theorem, see Proofs in Mathematics on page 211.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$f(-2) = 3(-2)^3 + 8(-2)^2 + 5(-2) - 7$$
$$= 3(-8) + 8(4) - 10 - 7 = -9$$

CHECKPoint Now try Exercise 55.

Another important theorem is the **Factor Theorem,** stated below. This theorem states that you can test to see whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

For a proof of the Factor Theorem, see Proofs in Mathematics on page 211.

Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$. Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

2	2	7	−4	−27	−18
		4	22	36	18
	2	11	18	9	0

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

−3	2	11	18	9
		−6	−15	−9
	2	5	3	0

$2x^2 + 5x + 3$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Graphical Solution

From the graph of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$, you can see that there are four x-intercepts (see Figure 2.29). These occur at $x = -3$, $x = -\frac{3}{2}$, $x = -1$, and $x = 2$. (Check this algebraically.) This implies that $(x + 3)$, $\left(x + \frac{3}{2}\right)$, $(x + 1)$, and $(x - 2)$ are factors of $f(x)$. $\left[\text{Note that } \left(x + \frac{3}{2}\right) \text{ and } (2x + 3) \text{ are equivalent factors because they both yield the same zero, } x = -\frac{3}{2}.\right]$

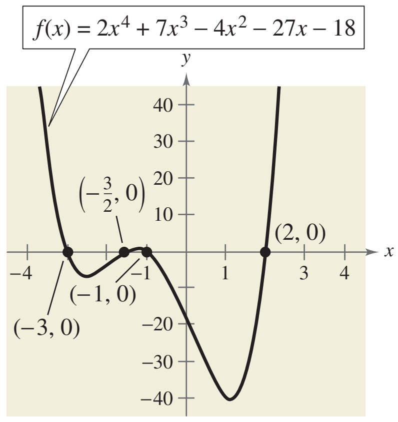


FIGURE 2.29

CHECK Point Now try Exercise 67.

Study Tip

Note in Example 6 that the complete factorization of $f(x)$ implies that f has four real zeros: $x = 2$, $x = -3$, $x = -\frac{3}{2}$, and $x = -1$. This is confirmed by the graph of f, which is shown in the Figure 2.29.

Uses of the Remainder in Synthetic Division

The remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$ (with no remainder), try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

2.3 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY

1. Two forms of the Division Algorithm are shown below. Identify and label each term or function.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–6, fill in the blanks.

2. The rational expression $p(x)/q(x)$ is called ________ if the degree of the numerator is greater than or equal to that of the denominator, and is called ________ if the degree of the numerator is less than that of the denominator.

3. In the Division Algorithm, the rational expression $f(x)/d(x)$ is ________ because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$.

4. An alternative method to long division of polynomials is called ________ ________, in which the divisor must be of the form $x - k$.

5. The ________ Theorem states that a polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

6. The ________ Theorem states that if a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

SKILLS AND APPLICATIONS

ANALYTICAL ANALYSIS In Exercises 7 and 8, use long division to verify that $y_1 = y_2$.

7. $y_1 = \dfrac{x^2}{x + 2}, \quad y_2 = x - 2 + \dfrac{4}{x + 2}$

8. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

GRAPHICAL ANALYSIS In Exercises 9 and 10, (a) use a graphing utility to graph the two equations in the same viewing window, (b) use the graphs to verify that the expressions are equivalent, and (c) use long division to verify the results algebraically.

9. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \dfrac{2}{x + 3}$

10. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \dfrac{1}{x^2 + 1}$

In Exercises 11–26, use long division to divide.

11. $(2x^2 + 10x + 12) \div (x + 3)$

12. $(5x^2 - 17x - 12) \div (x - 4)$

13. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$

14. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$

15. $(x^4 + 5x^3 + 6x^2 - x - 2) \div (x + 2)$

16. $(x^3 + 4x^2 - 3x - 12) \div (x - 3)$

17. $(x^3 - 27) \div (x - 3)$

18. $(x^3 + 125) \div (x + 5)$

19. $(7x + 3) \div (x + 2)$

20. $(8x - 5) \div (2x + 1)$

21. $(x^3 - 9) \div (x^2 + 1)$

22. $(x^5 + 7) \div (x^3 - 1)$

23. $(3x + 2x^3 - 9 - 8x^2) \div (x^2 + 1)$

24. $(5x^3 - 16 - 20x + x^4) \div (x^2 - x - 3)$

25. $\dfrac{x^4}{(x - 1)^3}$

26. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$

In Exercises 27–46, use synthetic division to divide.

27. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

28. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

29. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

30. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

31. $(4x^3 - 9x + 8x^2 - 18) \div (x + 2)$

32. $(9x^3 - 16x - 18x^2 + 32) \div (x - 2)$

33. $(-x^3 + 75x - 250) \div (x + 10)$

34. $(3x^3 - 16x^2 - 72) \div (x - 6)$

35. $(5x^3 - 6x^2 + 8) \div (x - 4)$

36. $(5x^3 + 6x + 8) \div (x + 2)$

37. $\dfrac{10x^4 - 50x^3 - 800}{x - 6}$

38. $\dfrac{x^5 - 13x^4 - 120x + 80}{x + 3}$

39. $\dfrac{x^3 + 512}{x + 8}$

40. $\dfrac{x^3 - 729}{x - 9}$

41. $\dfrac{-3x^4}{x - 2}$

42. $\dfrac{-3x^4}{x + 2}$

43. $\dfrac{180x - x^4}{x - 6}$

44. $\dfrac{5 - 3x + 2x^2 - x^3}{x + 1}$

45. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

46. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

In Exercises 47–54, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.

47. $f(x) = x^3 - x^2 - 14x + 11, \quad k = 4$

48. $f(x) = x^3 - 5x^2 - 11x + 8, \quad k = -2$

49. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14, \quad k = -\frac{2}{3}$

50. $f(x) = 10x^3 - 22x^2 - 3x + 4, \quad k = \frac{1}{5}$

51. $f(x) = x^3 + 3x^2 - 2x - 14, \quad k = \sqrt{2}$

52. $f(x) = x^3 + 2x^2 - 5x - 4, \quad k = -\sqrt{5}$

53. $f(x) = -4x^3 + 6x^2 + 12x + 4, \quad k = 1 - \sqrt{3}$

54. $f(x) = -3x^3 + 8x^2 + 10x - 8, \quad k = 2 + \sqrt{2}$

In Exercises 55–58, use the Remainder Theorem and synthetic division to find each function value. Verify your answers using another method.

55. $f(x) = 2x^3 - 7x + 3$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

56. $g(x) = 2x^6 + 3x^4 - x^2 + 3$

(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

57. $h(x) = x^3 - 5x^2 - 7x + 4$

(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

58. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$

(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 59–66, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all real solutions of the equation.

59. $x^3 - 7x + 6 = 0, \quad x = 2$

60. $x^3 - 28x - 48 = 0, \quad x = -4$

61. $2x^3 - 15x^2 + 27x - 10 = 0, \quad x = \frac{1}{2}$

62. $48x^3 - 80x^2 + 41x - 6 = 0, \quad x = \frac{2}{3}$

63. $x^3 + 2x^2 - 3x - 6 = 0, \quad x = \sqrt{3}$

64. $x^3 + 2x^2 - 2x - 4 = 0, \quad x = \sqrt{2}$

65. $x^3 - 3x^2 + 2 = 0, \quad x = 1 + \sqrt{3}$

66. $x^3 - x^2 - 13x - 3 = 0, \quad x = 2 - \sqrt{5}$

In Exercises 67–74, (a) verify the given factors of the function f, (b) find the remaining factor(s) of f, (c) use your results to write the complete factorization of f, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

	Function	*Factors*
67.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
68.	$f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3), (x - 2)$
69.	$f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
70.	$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
71.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
72.	$f(x) = 10x^3 - 11x^2 - 72x + 45$	$(2x + 5), (5x - 3)$
73.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), (x + \sqrt{5})$
74.	$f(x) = x^3 + 3x^2 - 48x - 144$	$(x + 4\sqrt{3}), (x + 3)$

GRAPHICAL ANALYSIS In Exercises 75–80, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine one of the exact zeros, and (c) use synthetic division to verify your result from part (b), and then factor the polynomial completely.

75. $f(x) = x^3 - 2x^2 - 5x + 10$

76. $g(x) = x^3 - 4x^2 - 2x + 8$

77. $h(t) = t^3 - 2t^2 - 7t + 2$

78. $f(s) = s^3 - 12s^2 + 40s - 24$

79. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

80. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 81–84, simplify the rational expression by using long division or synthetic division.

81. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$

82. $\dfrac{x^3 + x^2 - 64x - 64}{x + 8}$

83. $\dfrac{x^4 + 6x^3 + 11x^2 + 6x}{x^2 + 3x + 2}$

84. $\dfrac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4}$

85. DATA ANALYSIS: HIGHER EDUCATION The amounts A (in billions of dollars) donated to support higher education in the United States from 2000 through 2007 are shown in the table, where t represents the year, with $t = 0$ corresponding to 2000.

Year, t	Amount, A
0	23.2
1	24.2
2	23.9
3	23.9
4	24.4
5	25.6
6	28.0
7	29.8

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of A. Compare the model with the original data.

(d) Use synthetic division to evaluate the model for the year 2010. Even though the model is relatively accurate for estimating the given data, would you use this model to predict the amount donated to higher education in the future? Explain.

86. DATA ANALYSIS: HEALTH CARE The amounts A (in billions of dollars) of national health care expenditures in the United States from 2000 through 2007 are shown in the table, where t represents the year, with $t = 0$ corresponding to 2000.

Year, t	Amount, A
0	30.5
1	32.2
2	34.2
3	38.0
4	42.7
5	47.9
6	52.7
7	57.6

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of A. Compare the model with the original data.

(d) Use synthetic division to evaluate the model for the year 2010.

EXPLORATION

TRUE OR FALSE? In Exercises 87–89, determine whether the statement is true or false. Justify your answer.

87. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

88. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

89. The rational expression

$$\frac{x^3 + 2x^2 - 13x + 10}{x^2 - 4x - 12}$$

is improper.

90. Use the form $f(x) = (x - k)q(x) + r$ to create a cubic function that (a) passes through the point $(2, 5)$ and rises to the right, and (b) passes through the point $(-3, 1)$ and falls to the right. (There are many correct answers.)

THINK ABOUT IT In Exercises 91 and 92, perform the division by assuming that n is a positive integer.

91. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$ **92.** $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

93. WRITING Briefly explain what it means for a divisor to divide evenly into a dividend.

94. WRITING Briefly explain how to check polynomial division, and justify your reasoning. Give an example.

EXPLORATION In Exercises 95 and 96, find the constant c such that the denominator will divide evenly into the numerator.

95. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$ **96.** $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

97. THINK ABOUT IT Find the value of k such that $x - 4$ is a factor of $x^3 - kx^2 + 2kx - 8$.

98. THINK ABOUT IT Find the value of k such that $x - 3$ is a factor of $x^3 - kx^2 + 2kx - 12$.

99. WRITING Complete each polynomial division. Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

(a) $\dfrac{x^2 - 1}{x - 1} = \square$ (b) $\dfrac{x^3 - 1}{x - 1} = \square$

(c) $\dfrac{x^4 - 1}{x - 1} = \square$

100. CAPSTONE Consider the division

$f(x) \div (x - k)$

where

$f(x) = (x + 3)^2(x - 3)(x + 1)^3$.

(a) What is the remainder when $k = -3$? Explain.

(b) If it is necessary to find $f(2)$, it is easier to evaluate the function directly or to use synthetic division? Explain.

2.4 Complex Numbers

What you should learn

- Use the imaginary unit i to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

Why you should learn it

You can use complex numbers to model and solve real-life problems in electronics. For instance, in Exercise 89 on page 165, you will learn how to use complex numbers to find the impedance of an electrical circuit.

The Imaginary Unit *i*

You have learned that some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, the set of **complex numbers** is obtained. Each complex number can be written in the **standard form $a + bi$.** For instance, the standard form of the complex number $-5 + \sqrt{-9}$ is $-5 + 3i$ because

$$-5 + \sqrt{-9} = -5 + \sqrt{3^2(-1)} = -5 + 3\sqrt{-1} = -5 + 3i.$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number $a + bi$,** and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.30. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

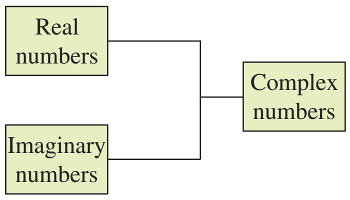


FIGURE 2.30

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$-(a + bi) = -a - bi.$ Additive inverse

So, you have

$(a + bi) + (-a - bi) = 0 + 0i = 0.$

Example 1 Adding and Subtracting Complex Numbers

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ Remove parentheses.

$= (4 + 1) + (7i - 6i)$ Group like terms.

$= 5 + i$ Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ Remove parentheses.

$= (1 - 4) + (2i - 2i)$ Group like terms.

$= -3 + 0$ Simplify.

$= -3$ Write in standard form.

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3i - 3i - 5i)$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

CHECK*Point* Now try Exercise 21.

Note in Examples 1(b) and 1(d) that the sum of two complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication Over Addition

Notice below how these properties are used when two complex numbers are multiplied.

$(a + bi)(c + di) = a(c + di) + bi(c + di)$ Distributive Property

$= ac + (ad)i + (bc)i + (bd)i^2$ Distributive Property

$= ac + (ad)i + (bc)i + (bd)(-1)$ $i^2 = -1$

$= ac - bd + (ad)i + (bc)i$ Commutative Property

$= (ac - bd) + (ad + bc)i$ Associative Property

Rather than trying to memorize this multiplication rule, you should simply remember how the Distributive Property is used to multiply two complex numbers.

Example 2 Multiplying Complex Numbers

a. $4(-2 + 3i) = 4(-2) + 4(3i)$ Distributive Property

$= -8 + 12i$ Simplify.

b. $(2 - i)(4 + 3i) = 2(4 + 3i) - i(4 + 3i)$ Distributive Property

$= 8 + 6i - 4i - 3i^2$ Distributive Property

$= 8 + 6i - 4i - 3(-1)$ $i^2 = -1$

$= (8 + 3) + (6i - 4i)$ Group like terms.

$= 11 + 2i$ Write in standard form.

c. $(3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)$ Distributive Property

$= 9 - 6i + 6i - 4i^2$ Distributive Property

$= 9 - 6i + 6i - 4(-1)$ $i^2 = -1$

$= 9 + 4$ Simplify.

$= 13$ Write in standard form.

d. $(3 + 2i)^2 = (3 + 2i)(3 + 2i)$ Square of a binomial

$= 3(3 + 2i) + 2i(3 + 2i)$ Distributive Property

$= 9 + 6i + 6i + 4i^2$ Distributive Property

$= 9 + 6i + 6i + 4(-1)$ $i^2 = -1$

$= 9 + 12i - 4$ Simplify.

$= 5 + 12i$ Write in standard form.

CHECKPoint Now try Exercise 31.

Study Tip

The procedure described above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method shown in Appendix A.3. For instance, you can use the FOIL Method to multiply the two complex numbers from Example 2(b).

$$(2 - i)(4 + 3i) = \overset{F}{8} + \overset{O}{6i} - \overset{I}{4i} - \overset{L}{3i^2}$$

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

Algebra Help

You can compare complex conjugates with the method for rationalizing denominators in Appendix A.2.

Example 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

CHECKPoint Now try Exercise 41.

Study Tip

Note that when you multiply the numerator and denominator of a quotient of complex numbers by

$$\frac{c - di}{c - di}$$

you are actually multiplying the quotient by a form of 1. You are not changing the original expression, you are only creating an expression that is equivalent to the original expression.

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}. &&\text{Standard form}\end{aligned}$$

Example 4 Writing a Quotient of Complex Numbers in Standard Form

$$\begin{aligned}\frac{2 + 3i}{4 - 2i} &= \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) &&\text{Multiply numerator and denominator by complex conjugate of denominator.} \\ &= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} &&\text{Expand.} \\ &= \frac{8 - 6 + 16i}{16 + 4} &&i^2 = -1 \\ &= \frac{2 + 16i}{20} &&\text{Simplify.} \\ &= \frac{1}{10} + \frac{4}{5}i &&\text{Write in standard form.}\end{aligned}$$

CHECKPoint Now try Exercise 53.

Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

Algebra Help

You can review the techniques for using the Quadratic Formula in Appendix A.5.

WARNING / CAUTION

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid if *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25}i^2\\ &= 5i^2 = -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

Example 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i$

c. $$\begin{aligned}\left(-1 + \sqrt{-3}\right)^2 &= \left(-1 + \sqrt{3}i\right)^2\\ &= (-1)^2 - 2\sqrt{3}i + \left(\sqrt{3}\right)^2(i^2)\\ &= 1 - 2\sqrt{3}i + 3(-1)\\ &= -2 - 2\sqrt{3}i\end{aligned}$$

CHECKPoint Now try Exercise 63.

Example 6 Complex Solutions of a Quadratic Equation

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

a. $x^2 + 4 = 0$ Write original equation.

$x^2 = -4$ Subtract 4 from each side.

$x = \pm 2i$ Extract square roots.

b. $3x^2 - 2x + 5 = 0$ Write original equation.

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$ Quadratic Formula

$= \dfrac{2 \pm \sqrt{-56}}{6}$ Simplify.

$= \dfrac{2 \pm 2\sqrt{14}i}{6}$ Write $\sqrt{-56}$ in standard form.

$= \dfrac{1}{3} \pm \dfrac{\sqrt{14}}{3}i$ Write in standard form.

CHECKPoint Now try Exercise 69.

2.4 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY

1. Match the type of complex number with its definition.

(a) Real number (i) $a + bi,\quad a \neq 0,\quad b \neq 0$

(b) Imaginary number (ii) $a + bi,\quad a = 0,\quad b \neq 0$

(c) Pure imaginary number (iii) $a + bi,\quad b = 0$

In Exercises 2–4, fill in the blanks.

2. The imaginary unit i is defined as $i =$ ________, where $i^2 =$ ________.

3. If a is a positive number, the ________ ________ root of the negative number $-a$ is defined as $\sqrt{-a} = \sqrt{a}\,i$.

4. The numbers $a + bi$ and $a - bi$ are called ________ ________, and their product is a real number $a^2 + b^2$.

SKILLS AND APPLICATIONS

In Exercises 5–8, find real numbers a and b such that the equation is true.

5. $a + bi = -12 + 7i$ **6.** $a + bi = 13 + 4i$

7. $(a - 1) + (b + 3)i = 5 + 8i$

8. $(a + 6) + 2bi = 6 - 5i$

In Exercises 9–20, write the complex number in standard form.

9. $8 + \sqrt{-25}$ **10.** $5 + \sqrt{-36}$

11. $2 - \sqrt{-27}$ **12.** $1 + \sqrt{-8}$

13. $\sqrt{-80}$ **14.** $\sqrt{-4}$

15. 14 **16.** 75

17. $-10i + i^2$ **18.** $-4i^2 + 2i$

19. $\sqrt{-0.09}$ **20.** $\sqrt{-0.0049}$

In Exercises 21–30, perform the addition or subtraction and write the result in standard form.

21. $(7 + i) + (3 - 4i)$ **22.** $(13 - 2i) + (-5 + 6i)$

23. $(9 - i) - (8 - i)$ **24.** $(3 + 2i) - (6 + 13i)$

25. $\left(-2 + \sqrt{-8}\right) + \left(5 - \sqrt{-50}\right)$

26. $\left(8 + \sqrt{-18}\right) - \left(4 + 3\sqrt{2}i\right)$

27. $13i - (14 - 7i)$

28. $25 + (-10 + 11i) + 15i$

29. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

30. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

In Exercises 31–40, perform the operation and write the result in standard form.

31. $(1 + i)(3 - 2i)$ **32.** $(7 - 2i)(3 - 5i)$

33. $12i(1 - 9i)$ **34.** $-8i(9 + 4i)$

35. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

36. $\left(\sqrt{3} + \sqrt{15}i\right)\left(\sqrt{3} - \sqrt{15}i\right)$

37. $(6 + 7i)^2$ **38.** $(5 - 4i)^2$

39. $(2 + 3i)^2 + (2 - 3i)^2$ **40.** $(1 - 2i)^2 - (1 + 2i)^2$

In Exercises 41–48, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

41. $9 + 2i$ **42.** $8 - 10i$

43. $-1 - \sqrt{5}i$ **44.** $-3 + \sqrt{2}i$

45. $\sqrt{-20}$ **46.** $\sqrt{-15}$

47. $\sqrt{6}$ **48.** $1 + \sqrt{8}$

In Exercises 49–58, write the quotient in standard form.

49. $\dfrac{3}{i}$ **50.** $-\dfrac{14}{2i}$

51. $\dfrac{2}{4 - 5i}$ **52.** $\dfrac{13}{1 - i}$

53. $\dfrac{5 + i}{5 - i}$ **54.** $\dfrac{6 - 7i}{1 - 2i}$

55. $\dfrac{9 - 4i}{i}$ **56.** $\dfrac{8 + 16i}{2i}$

57. $\dfrac{3i}{(4 - 5i)^2}$ **58.** $\dfrac{5i}{(2 + 3i)^2}$

In Exercises 59–62, perform the operation and write the result in standard form.

59. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$

60. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

61. $\dfrac{i}{3 - 2i} + \dfrac{2i}{3 + 8i}$

62. $\dfrac{1 + i}{i} - \dfrac{3}{4 - i}$

In Exercises 63–68, write the complex number in standard form.

63. $\sqrt{-6} \cdot \sqrt{-2}$

64. $\sqrt{-5} \cdot \sqrt{-10}$

65. $\left(\sqrt{-15}\right)^2$

66. $\left(\sqrt{-75}\right)^2$

67. $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

68. $\left(2 - \sqrt{-6}\right)^2$

In Exercises 69–78, use the Quadratic Formula to solve the quadratic equation.

69. $x^2 - 2x + 2 = 0$

70. $x^2 + 6x + 10 = 0$

71. $4x^2 + 16x + 17 = 0$

72. $9x^2 - 6x + 37 = 0$

73. $4x^2 + 16x + 15 = 0$

74. $16t^2 - 4t + 3 = 0$

75. $\frac{3}{2}x^2 - 6x + 9 = 0$

76. $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$

77. $1.4x^2 - 2x - 10 = 0$

78. $4.5x^2 - 3x + 12 = 0$

In Exercises 79–88, simplify the complex number and write it in standard form.

79. $-6i^3 + i^2$

80. $4i^2 - 2i^3$

81. $-14i^5$

82. $(-i)^3$

83. $\left(\sqrt{-72}\right)^3$

84. $\left(\sqrt{-2}\right)^6$

85. $\dfrac{1}{i^3}$

86. $\dfrac{1}{(2i)^3}$

87. $(3i)^4$

88. $(-i)^6$

89. IMPEDANCE The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation

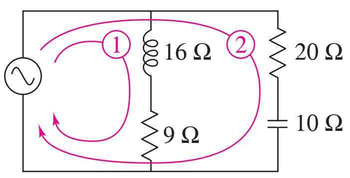
$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$

where z_1 is the impedance (in ohms) of pathway 1 and z_2 is the impedance of pathway 2.

(a) The impedance of each pathway in a parallel circuit is found by adding the impedances of all components in the pathway. Use the table to find z_1 and z_2.

(b) Find the impedance z.

	Resistor	Inductor	Capacitor
Symbol	$a\Omega$	$b\Omega$	$c\Omega$
Impedance	a	bi	$-ci$



90. Cube each complex number.

(a) 2 (b) $-1 + \sqrt{3}i$ (c) $-1 - \sqrt{3}i$

91. Raise each complex number to the fourth power.

(a) 2 (b) -2 (c) $2i$ (d) $-2i$

92. Write each of the powers of i as i, $-i$, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

EXPLORATION

TRUE OR FALSE? In Exercises 93–96, determine whether the statement is true or false. Justify your answer.

93. There is no complex number that is equal to its complex conjugate.

94. $-i\sqrt{6}$ is a solution of $x^4 - x^2 + 14 = 56$.

95. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

96. The sum of two complex numbers is always a real number.

97. PATTERN RECOGNITION Complete the following.

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$
$i^5 =$	$i^6 =$	$i^7 =$	$i^8 =$
$i^9 =$	$i^{10} =$	$i^{11} =$	$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

98. CAPSTONE Consider the functions

$f(x) = 2(x - 3)^2 - 4$ and $g(x) = -2(x - 3)^2 - 4$.

(a) Without graphing either function, determine whether the graph of f and the graph of g have x-intercepts. Explain your reasoning.

(b) Solve $f(x) = 0$ and $g(x) = 0$.

(c) Explain how the zeros of f and g are related to whether their graphs have x-intercepts.

(d) For the function $f(x) = a(x - h)^2 + k$, make a general statement about how a, h, and k affect whether the graph of f has x-intercepts, and whether the zeros of f are real or complex.

99. ERROR ANALYSIS Describe the error.

$$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6$$

100. PROOF Prove that the complex conjugate of the product of two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is the product of their complex conjugates.

101. PROOF Prove that the complex conjugate of the sum of two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is the sum of their complex conjugates.

2.5 Zeros of Polynomial Functions

What you should learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions.
- Find rational zeros of polynomial functions.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

Finding zeros of polynomial functions is an important part of solving real-life problems. For instance, in Exercise 120 on page 179, the zeros of a polynomial function can help you analyze the attendance at women's college basketball games.

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

> **The Fundamental Theorem of Algebra**
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

> **Linear Factorization Theorem**
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors
>
> $$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$
>
> where $c_1, c_2, . . . , c_n$ are complex numbers.

For a proof of the Linear Factorization Theorem, see Proofs in Mathematics on page 212.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called *existence theorems*. Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated.

Study Tip

Recall that in order to find the zeros of a function $f(x)$, set $f(x)$ equal to 0 and solve the resulting equation for x. For instance, the function in Example 1(a) has a zero at $x = 2$ because

$$x - 2 = 0$$
$$x = 2.$$

Algebra Help

Examples 1(b), 1(c), and 1(d) involve factoring polynomials. You can review the techniques for factoring polynomials in Appendix A.3.

Example 1 Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function

$$f(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

CHECKPoint Now try Exercise 9.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

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The Rational Zero Test

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

p = a factor of the constant term a_0

q = a factor of the leading coefficient a_n.

To use the Rational Zero Test, you should first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Having formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Example 2 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^3 + x + 1.$$

Solution

Because the leading coefficient is 1, the possible rational zeros are ± 1, the factors of the constant term. By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1$$
$$= 3$$
$$f(-1) = (-1)^3 + (-1) + 1$$
$$= -1$$

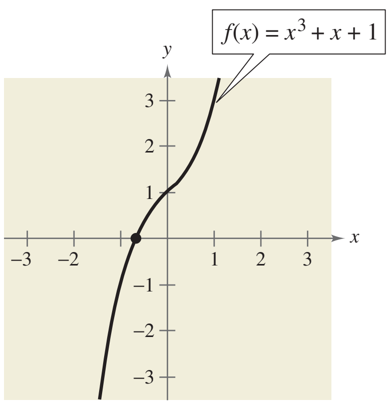


FIGURE 2.31

So, you can conclude that the given polynomial has *no* rational zeros. Note from the graph of f in Figure 2.31 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

CHECKPoint Now try Exercise 15.

Example 3 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^4 - x^3 + x^2 - 3x - 6$.

Solution

Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

By applying synthetic division successively, you can determine that $x = -1$ and $x = 2$ are the only two rational zeros.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array}$$

0 remainder, so $x = -1$ is a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

0 remainder, so $x = 2$ is a zero.

So, $f(x)$ factors as

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, you can conclude that $x = -1$ and $x = 2$ are the only *real* zeros of f, which is verified in Figure 2.32.

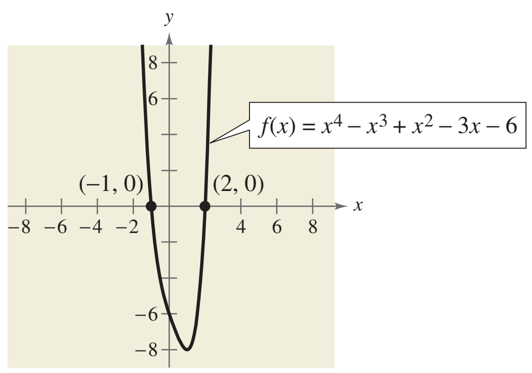


FIGURE 2.32

CHECKPoint Now try Exercise 19.

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways: (1) a programmable calculator can be used to speed up the calculations; (2) a graph, drawn either by hand or with a graphing utility, can give a good estimate of the locations of the zeros; (3) the Intermediate Value Theorem along with a table generated by a graphing utility can give approximations of zeros; and (4) synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 3.

Study Tip

When the list of possible rational zeros is small, as in Example 2, it may be quicker to test the zeros by evaluating the function. When the list of possible rational zeros is large, as in Example 3, it may be quicker to use a different approach to test the zeros, such as using synthetic division or sketching a graph.

Algebra Help

You can review the techniques for synthetic division in Section 2.3.

Example 4 Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3)$$
$$= (x - 1)(2x - 1)(x + 3)$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

CHECKPoint Now try Exercise 25.

> **Study Tip**
>
> Remember that when you try to find the rational zeros of a polynomial function with many possible rational zeros, as in Example 4, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

Recall from Section 2.2 that if $x = a$ is a zero of the polynomial function f, then $x = a$ is a solution of the polynomial equation $f(x) = 0$.

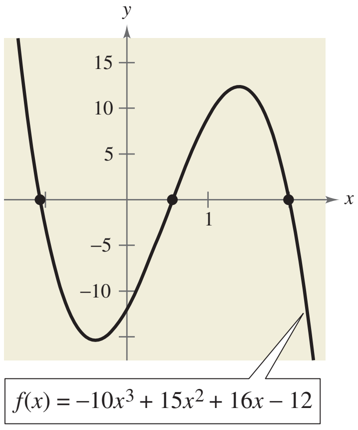


FIGURE 2.33

Example 5 Solving a Polynomial Equation

Find all the real solutions of $-10x^3 + 15x^2 + 16x - 12 = 0$.

Solution

The leading coefficient is -10 and the constant term is -12.

$$\textit{Possible rational solutions: } \frac{\text{Factors of } -12}{\text{Factors of } -10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to stop and sketch a graph. From Figure 2.33, it looks like three reasonable solutions would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that $x = 2$ is the only rational solution. So, you have

$$(x - 2)(-10x^2 - 5x + 6) = 0.$$

Using the Quadratic Formula for the second factor, you find that the two additional solutions are irrational numbers.

$$x = \frac{-5 - \sqrt{265}}{20} \approx -1.0639$$

and

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.5639$$

CHECKPoint Now try Exercise 31.

> **Algebra Help**
>
> You can review the techniques for using the Quadratic Formula in Appendix A.5.

Conjugate Pairs

In Examples 1(c) and 1(d), note that the pairs of complex zeros are **conjugates.** That is, they are of the form $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients*. For instance, the result applies to the function given by $f(x) = x^2 + 1$ but not to the function given by $g(x) = x - i$.

Example 6 Finding a Polynomial with Given Zeros

Find a fourth-degree polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

CHECKPoint Now try Exercise 45.

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following theorem says that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors. For a proof of this theorem, see Proofs in Mathematics on page 212.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be *prime* or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic $x^2 + 1 = (x - i)(x + i)$ is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ is irreducible over the rationals but *reducible* over the reals.

Example 7 Finding the Zeros of a Polynomial Function

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10 \overline{\smash{)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \qquad\qquad \\ -x^3 - 4x^2 + 2x \qquad \\ \underline{-x^3 + 2x^2 - 10x} \qquad \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

Graphical Solution

Because complex zeros always occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. Because the polynomial is a fourth-degree polynomial, you know that there are at most two other zeros of the function. Use a graphing utility to graph

$$y = x^4 - 3x^3 + 6x^2 + 2x - 60$$

as shown in Figure 2.34.

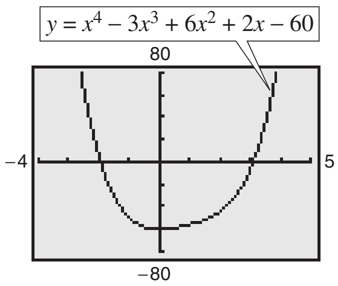


FIGURE 2.34

You can see that -2 and 3 appear to be zeros of the graph of the function. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to confirm that $x = -2$ and $x = 3$ are zeros of the graph. So, you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

CHECKPoint Now try Exercise 55.

Algebra Help

You can review the techniques for polynomial long division in Section 2.3.

In Example 7, if you were not told that $1 + 3i$ is a zero of f, you could still find all zeros of the function by using synthetic division to find the real zeros -2 and 3. Then you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = -2$, $x = 3$, $x = 1 + 3i$, and $x = 1 - 3i$.

Study Tip

In Example 8, the fifth-degree polynomial function has three real zeros. In such cases, you can use the *zoom* and *trace* features or the *zero* or *root* feature of a graphing utility to approximate the real zeros. You can then use these real zeros to determine the complex zeros algebraically.

Example 8 shows how to find all the zeros of a polynomial function, including complex zeros.

Example 8 Finding the Zeros of a Polynomial Function

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all of its zeros.

Solution

The possible rational zeros are $\pm1, \pm2, \pm4,$ and ±8. Synthetic division produces the following.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array} \quad \text{1 is a zero.}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & 2 & 4 & -8 \\ & & -2 & 2 & -8 & 8 \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array} \quad -2 \text{ is a zero.}$$

So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4). \end{aligned}$$

You can factor $x^3 - x^2 + 4x - 4$ as $(x - 1)(x^2 + 4)$, and by factoring $x^2 + 4$ as

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1,\ x = 1,\ x = -2,\ x = 2i, \quad \text{and} \quad x = -2i$$

From the graph of f shown in Figure 2.35, you can see that the *real* zeros are the only ones that appear as x-intercepts. Note that $x = 1$ is a repeated zero.

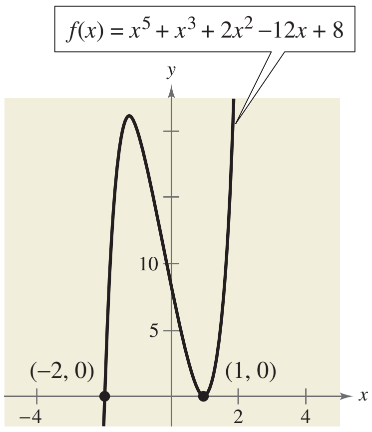
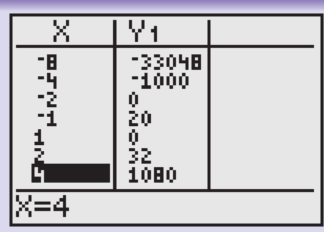


FIGURE 2.35

CHECKPoint Now try Exercise 77.

TECHNOLOGY

You can use the *table* feature of a graphing utility to help you determine which of the possible rational zeros are zeros of the polynomial in Example 8. The table should be set to *ask* mode. Then enter each of the possible rational zeros in the table. When you do this, you will see that there are two rational zeros, −2 and 1, as shown at the right.



Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 9 Using Descartes's Rule of Signs

Describe the possible real zeros of

$$f(x) = 3x^3 - 5x^2 + 6x - 4.$$

Solution

The original polynomial has *three* variations in sign.

+ to − + to −

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

− to +

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. From the graph in Figure 2.36, you can see that the function has only one real zero, at $x = 1$.

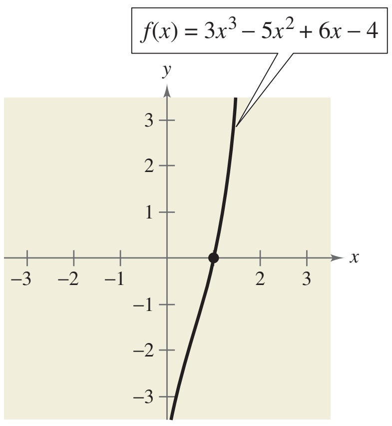


FIGURE 2.36

CHECKPoint Now try Exercise 87.

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f. A real number b is an **upper bound** for the real zeros of f if no zeros are greater than b. Similarly, b is a **lower bound** if no real zeros of f are less than b.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

Example 10 Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$\begin{aligned} f(-x) &= 6(-x)^3 - 4(-x)^2 + 3(-x) - 2 \\ &= -6x^3 - 4x^2 - 3x - 2 \end{aligned}$$

has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. So, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero.

CHECKPoint Now try Exercise 95.

Before concluding this section, here are two additional hints that can help you find the real zeros of a polynomial.

1. If the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$\begin{aligned} f(x) &= x^4 - 5x^3 + 3x^2 + x \\ &= x(x^3 - 5x^2 + 3x + 1) \end{aligned}$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. If you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

$$\begin{aligned} f(x) &= x^4 - 5x^3 + 3x^2 + x \\ &= x(x - 1)(x^2 - 4x - 1) \end{aligned}$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Example 11 Using a Polynomial Model

You are designing candle-making kits. Each kit contains 25 cubic inches of candle wax and a mold for making a pyramid-shaped candle. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?

Solution

The volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height. The area of the base is x^2 and the height is $(x - 2)$. So, the volume of the pyramid is $V = \frac{1}{3}x^2(x - 2)$. Substituting 25 for the volume yields the following.

$$25 = \frac{1}{3}x^2(x - 2) \qquad \text{Substitute 25 for } V.$$

$$75 = x^3 - 2x^2 \qquad \text{Multiply each side by 3.}$$

$$0 = x^3 - 2x^2 - 75 \qquad \text{Write in general form.}$$

The possible rational solutions are $x = \pm1, \pm3, \pm5, \pm15, \pm25, \pm75$. Use synthetic division to test some of the possible solutions. Note that in this case, it makes sense to test only positive x-values. Using synthetic division, you can determine that $x = 5$ is a solution.

$$\begin{array}{r|rrrr} 5 & 1 & -2 & 0 & -75 \\ & & 5 & 15 & 75 \\ \hline & 1 & 3 & 15 & 0 \end{array}$$

The other two solutions, which satisfy $x^2 + 3x + 15 = 0$, are imaginary and can be discarded. You can conclude that the base of the candle mold should be 5 inches by 5 inches and the height of the mold should be $5 - 2 = 3$ inches.

CHECK*Point* Now try Exercise 115.

2.5 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. The ________ ________ of ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then f has at least one zero in the complex number system.
2. The ________ ________ ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then f has precisely n linear factors, $f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$, where $c_1, c_2, . . . , c_n$ are complex numbers.
3. The test that gives a list of the possible rational zeros of a polynomial function is called the ________ ________ Test.
4. If $a + bi$ is a complex zero of a polynomial with real coefficients, then so is its ________, $a - bi$.
5. Every polynomial of degree $n > 0$ with real coefficients can be written as the product of ________ and ________ factors with real coefficients, where the ________ factors have no real zeros.
6. A quadratic factor that cannot be factored further as a product of linear factors containing real numbers is said to be ________ over the ________.
7. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called ________ ________ of ________.
8. A real number b is a(n) ________ bound for the real zeros of f if no real zeros are less than b, and is a(n) ________ bound if no real zeros are greater than b.

SKILLS AND APPLICATIONS

In Exercises 9–14, find all the zeros of the function.

9. $f(x) = x(x - 6)^2$

10. $f(x) = x^2(x + 3)(x^2 - 1)$

11. $g(x) = (x - 2)(x + 4)^3$

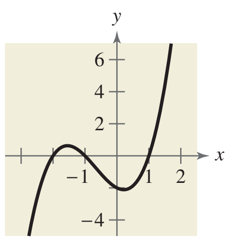
12. $f(x) = (x + 5)(x - 8)^2$

13. $f(x) = (x + 6)(x + i)(x - i)$

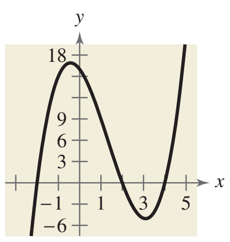
14. $h(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$

In Exercises 15–18, use the Rational Zero Test to list all possible rational zeros of f. Verify that the zeros of f shown on the graph are contained in the list.

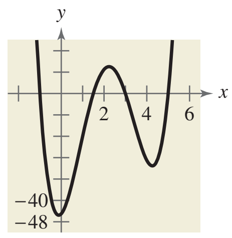
15. $f(x) = x^3 + 2x^2 - x - 2$



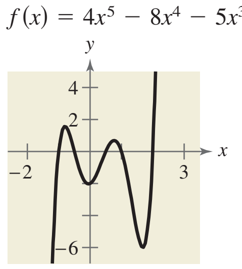
16. $f(x) = x^3 - 4x^2 - 4x + 16$



17. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$



18. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$



In Exercises 19–28, find all the rational zeros of the function.

19. $f(x) = x^3 - 6x^2 + 11x - 6$

20. $f(x) = x^3 - 7x - 6$

21. $g(x) = x^3 - 4x^2 - x + 4$

22. $h(x) = x^3 - 9x^2 + 20x - 12$

23. $h(t) = t^3 + 8t^2 + 13t + 6$

24. $p(x) = x^3 - 9x^2 + 27x - 27$

25. $C(x) = 2x^3 + 3x^2 - 1$

26. $f(x) = 3x^3 - 19x^2 + 33x - 9$

27. $f(x) = 9x^4 - 9x^3 - 58x^2 + 4x + 24$

28. $f(x) = 2x^4 - 15x^3 + 23x^2 + 15x - 25$

In Exercises 29–32, find all real solutions of the polynomial equation.

29. $z^4 + z^3 + z^2 + 3z - 6 = 0$

30. $x^4 - 13x^2 - 12x = 0$

31. $2y^4 + 3y^3 - 16y^2 + 15y - 4 = 0$

32. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

In Exercises 33–36, (a) list the possible rational zeros of f, (b) sketch the graph of f so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of f.

33. $f(x) = x^3 + x^2 - 4x - 4$

34. $f(x) = -3x^3 + 20x^2 - 36x + 16$

35. $f(x) = -4x^3 + 15x^2 - 8x - 3$

36. $f(x) = 4x^3 - 12x^2 - x + 15$

In Exercises 37–40, (a) list the possible rational zeros of f, (b) use a graphing utility to graph f so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of f.

37. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

38. $f(x) = 4x^4 - 17x^2 + 4$

39. $f(x) = 32x^3 - 52x^2 + 17x + 3$

40. $f(x) = 4x^3 + 7x^2 - 11x - 18$

GRAPHICAL ANALYSIS In Exercises 41–44, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine one of the exact zeros (use synthetic division to verify your result), and (c) factor the polynomial completely.

41. $f(x) = x^4 - 3x^2 + 2$

42. $P(t) = t^4 - 7t^2 + 12$

43. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

44. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 45–50, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

45. $1, 5i$

46. $4, -3i$

47. $2, 5 + i$

48. $5, 3 - 2i$

49. $\frac{2}{3}, -1, 3 + \sqrt{2}i$

50. $-5, -5, 1 + \sqrt{3}i$

In Exercises 51–54, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

51. $f(x) = x^4 + 6x^2 - 27$

52. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$
(*Hint:* One factor is $x^2 - 6$.)

53. $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$
(*Hint:* One factor is $x^2 - 2x - 2$.)

54. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$
(*Hint:* One factor is $x^2 + 4$.)

In Exercises 55–62, use the given zero to find all the zeros of the function.

	Function	*Zero*
55.	$f(x) = x^3 - x^2 + 4x - 4$	$2i$
56.	$f(x) = 2x^3 + 3x^2 + 18x + 27$	$3i$
57.	$f(x) = 2x^4 - x^3 + 49x^2 - 25x - 25$	$5i$
58.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
59.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
60.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
61.	$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$	$-3 + \sqrt{2}i$
62.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$

In Exercises 63–80, find all the zeros of the function and write the polynomial as a product of linear factors.

63. $f(x) = x^2 + 36$

64. $f(x) = x^2 - x + 56$

65. $h(x) = x^2 - 2x + 17$

66. $g(x) = x^2 + 10x + 17$

67. $f(x) = x^4 - 16$

68. $f(y) = y^4 - 256$

69. $f(z) = z^2 - 2z + 2$

70. $h(x) = x^3 - 3x^2 + 4x - 2$

71. $g(x) = x^3 - 3x^2 + x + 5$

72. $f(x) = x^3 - x^2 + x + 39$

73. $h(x) = x^3 - x + 6$

74. $h(x) = x^3 + 9x^2 + 27x + 35$

75. $f(x) = 5x^3 - 9x^2 + 28x + 6$

76. $g(x) = 2x^3 - x^2 + 8x + 21$

77. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

78. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

79. $f(x) = x^4 + 10x^2 + 9$

80. $f(x) = x^4 + 29x^2 + 100$

In Exercises 81–86, find all the zeros of the function. When there is an extended list of possible rational zeros, use a graphing utility to graph the function in order to discard any rational zeros that are obviously not zeros of the function.

81. $f(x) = x^3 + 24x^2 + 214x + 740$

82. $f(s) = 2s^3 - 5s^2 + 12s - 5$

83. $f(x) = 16x^3 - 20x^2 - 4x + 15$

84. $f(x) = 9x^3 - 15x^2 + 11x - 5$

85. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$

86. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

In Exercises 87–94, use Descartes's Rule of Signs to determine the possible numbers of positive and negative zeros of the function.

87. $g(x) = 2x^3 - 3x^2 - 3$ **88.** $h(x) = 4x^2 - 8x + 3$

89. $h(x) = 2x^3 + 3x^2 + 1$ **90.** $h(x) = 2x^4 - 3x + 2$

91. $g(x) = 5x^5 - 10x$

92. $f(x) = 4x^3 - 3x^2 + 2x - 1$

93. $f(x) = -5x^3 + x^2 - x + 5$

94. $f(x) = 3x^3 + 2x^2 + x + 3$

In Exercises 95–98, use synthetic division to verify the upper and lower bounds of the real zeros of *f*.

95. $f(x) = x^3 + 3x^2 - 2x + 1$

(a) Upper: $x = 1$ (b) Lower: $x = -4$

96. $f(x) = x^3 - 4x^2 + 1$

(a) Upper: $x = 4$ (b) Lower: $x = -1$

97. $f(x) = x^4 - 4x^3 + 16x - 16$

(a) Upper: $x = 5$ (b) Lower: $x = -3$

98. $f(x) = 2x^4 - 8x + 3$

(a) Upper: $x = 3$ (b) Lower: $x = -4$

In Exercises 99–102, find all the real zeros of the function.

99. $f(x) = 4x^3 - 3x - 1$

100. $f(z) = 12z^3 - 4z^2 - 27z + 9$

101. $f(y) = 4y^3 + 3y^2 + 8y + 6$

102. $g(x) = 3x^3 - 2x^2 + 15x - 10$

In Exercises 103–106, find all the rational zeros of the polynomial function.

103. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

104. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

105. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

106. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

In Exercises 107–110, match the cubic function with the numbers of rational and irrational zeros.

(a) Rational zeros: 0; irrational zeros: 1

(b) Rational zeros: 3; irrational zeros: 0

(c) Rational zeros: 1; irrational zeros: 2

(d) Rational zeros: 1; irrational zeros: 0

107. $f(x) = x^3 - 1$ **108.** $f(x) = x^3 - 2$

109. $f(x) = x^3 - x$ **110.** $f(x) = x^3 - 2x$

111. GEOMETRY An open box is to be made from a rectangular piece of material, 15 centimeters by 9 centimeters, by cutting equal squares from the corners and turning up the sides.

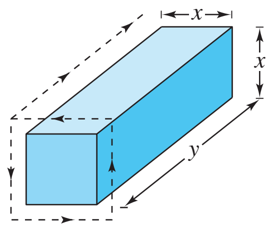
(a) Let x represent the length of the sides of the squares removed. Draw a diagram showing the squares removed from the original piece of material and the resulting dimensions of the open box.

(b) Use the diagram to write the volume V of the box as a function of x. Determine the domain of the function.

(c) Sketch the graph of the function and approximate the dimensions of the box that will yield a maximum volume.

(d) Find values of x such that $V = 56$. Which of these values is a physical impossibility in the construction of the box? Explain.

112. GEOMETRY A rectangular package to be sent by a delivery service (see figure) can have a maximum combined length and girth (perimeter of a cross section) of 120 inches.



(a) Write a function $V(x)$ that represents the volume of the package.

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that will yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

113. ADVERTISING COST A company that produces MP3 players estimates that the profit P (in dollars) for selling a particular model is given by

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, find the smaller of two advertising amounts that will yield a profit of \$2,500,000.

114. ADVERTISING COST A company that manufactures bicycles estimates that the profit P (in dollars) for selling a particular model is given by

$$P = -45x^3 + 2500x^2 - 275{,}000, \quad 0 \le x \le 50$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, find the smaller of two advertising amounts that will yield a profit of \$800,000.

115. GEOMETRY A bulk food storage bin with dimensions 2 feet by 3 feet by 4 feet needs to be increased in size to hold five times as much food as the current bin. (Assume each dimension is increased by the same amount.)

(a) Write a function that represents the volume V of the new bin.

(b) Find the dimensions of the new bin.

116. GEOMETRY A manufacturer wants to enlarge an existing manufacturing facility such that the total floor area is 1.5 times that of the current facility. The floor area of the current facility is rectangular and measures 250 feet by 160 feet. The manufacturer wants to increase each dimension by the same amount.

(a) Write a function that represents the new floor area A.

(b) Find the dimensions of the new floor.

(c) Another alternative is to increase the current floor's length by an amount that is twice an increase in the floor's width. The total floor area is 1.5 times that of the current facility. Repeat parts (a) and (b) using these criteria.

117. COST The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). In calculus, it can be shown that the cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Use a calculator to approximate the optimal order size to the nearest hundred units.

118. HEIGHT OF A BASEBALL A baseball is thrown upward from a height of 6 feet with an initial velocity of 48 feet per second, and its height h (in feet) is

$$h(t) = -16t^2 + 48t + 6, \quad 0 \leq t \leq 3$$

where t is the time (in seconds). You are told the ball reaches a height of 64 feet. Is this possible?

119. PROFIT The demand equation for a certain product is $p = 140 - 0.0001x$, where p is the unit price (in dollars) of the product and x is the number of units produced and sold. The cost equation for the product is $C = 80x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is $P = R - C = xp - C$. You are working in the marketing department of the company that produces this product, and you are asked to determine a price p that will yield a profit of 9 million dollars. Is this possible? Explain.

120. ATHLETICS The attendance A (in millions) at NCAA women's college basketball games for the years 2000 through 2007 is shown in the table. (Source: National Collegiate Athletic Association, Indianapolis, IN)

Year	Attendance, A
2000	8.7
2001	8.8
2002	9.5
2003	10.2
2004	10.0
2005	9.9
2006	9.9
2007	10.9

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a quartic model for the data.

(c) Graph the model and the scatter plot in the same viewing window. How well does the model fit the data?

(d) According to the model in part (b), in what year(s) was the attendance at least 10 million?

(e) According to the model, will the attendance continue to increase? Explain.

EXPLORATION

TRUE OR FALSE? In Exercises 121 and 122, decide whether the statement is true or false. Justify your answer.

121. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

122. If $x = -i$ is a zero of the function given by

$$f(x) = x^3 + ix^2 + ix - 1$$

then $x = i$ must also be a zero of f.

THINK ABOUT IT In Exercises 123–128, determine (if possible) the zeros of the function g if the function f has zeros at $x = r_1$, $x = r_2$, and $x = r_3$.

123. $g(x) = -f(x)$

124. $g(x) = 3f(x)$

125. $g(x) = f(x - 5)$

126. $g(x) = f(2x)$

127. $g(x) = 3 + f(x)$

128. $g(x) = f(-x)$

129. THINK ABOUT IT A third-degree polynomial function f has real zeros -2, $\frac{1}{2}$, and 3, and its leading coefficient is negative. Write an equation for f. Sketch the graph of f. How many different polynomial functions are possible for f?

130. CAPSTONE Use a graphing utility to graph the function given by $f(x) = x^4 - 4x^2 + k$ for different values of k. Find values of k such that the zeros of f satisfy the specified characteristics. (Some parts do not have unique answers.)

(a) Four real zeros

(b) Two real zeros, each of multiplicity 2

(c) Two real zeros and two complex zeros

(d) Four complex zeros

(e) Will the answers to parts (a) through (d) change for the function g, where $g(x) = f(x - 2)$?

(f) Will the answers to parts (a) through (d) change for the function g, where $g(x) = f(2x)$?

131. THINK ABOUT IT Sketch the graph of a fifth-degree polynomial function whose leading coefficient is positive and that has a zero at $x = 3$ of multiplicity 2.

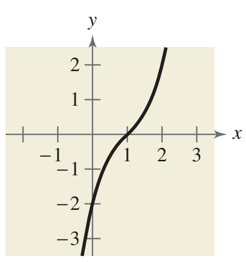
132. WRITING Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

133. THINK ABOUT IT Let $y = f(x)$ be a quartic polynomial with leading coefficient $a = 1$ and $f(i) = f(2i) = 0$. Write an equation for f.

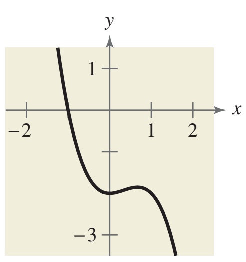
134. THINK ABOUT IT Let $y = f(x)$ be a cubic polynomial with leading coefficient $a = -1$ and $f(2) = f(i) = 0$. Write an equation for f.

In Exercises 135 and 136, the graph of a cubic polynomial function $y = f(x)$ is shown. It is known that one of the zeros is $1 + i$. Write an equation for f.

135.



136.



137. Use the information in the table to answer each question.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Write an equation for f. (There are many correct answers.)

(f) Sketch a graph of the equation you wrote in part (e).

138. (a) Find a quadratic function f (with integer coefficients) that has $\pm\sqrt{b}i$ as zeros. Assume that b is a positive integer.

(b) Find a quadratic function f (with integer coefficients) that has $a \pm bi$ as zeros. Assume that b is a positive integer.

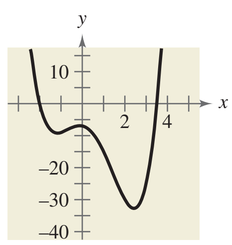
139. GRAPHICAL REASONING The graph of one of the following functions is shown below. Identify the function shown in the graph. Explain why each of the others is not the correct function. Use a graphing utility to verify your result.

(a) $f(x) = x^2(x + 2)(x - 3.5)$

(b) $g(x) = (x + 2)(x - 3.5)$

(c) $h(x) = (x + 2)(x - 3.5)(x^2 + 1)$

(d) $k(x) = (x + 1)(x + 2)(x - 3.5)$



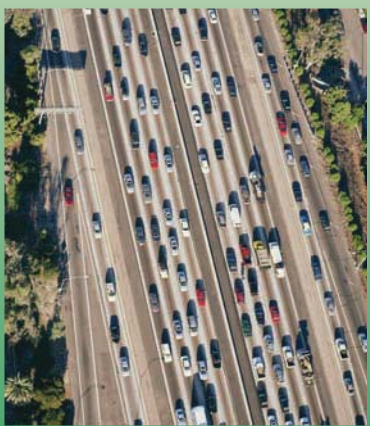
2.6 Rational Functions

What you should learn

- Find the domains of rational functions.
- Find the vertical and horizontal asymptotes of graphs of rational functions.
- Analyze and sketch graphs of rational functions.
- Sketch graphs of rational functions that have slant asymptotes.
- Use rational functions to model and solve real-life problems.

Why you should learn it

Rational functions can be used to model and solve real-life problems relating to business. For instance, in Exercise 83 on page 193, a rational function is used to model average speed over a distance.



Mike Powell/Getty Images

Introduction

A **rational function** is a quotient of polynomial functions. It can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near the x-values excluded from the domain.

Example 1 Finding the Domain of a Rational Function

Find the domain of the reciprocal function $f(x) = \frac{1}{x}$ and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\longrightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\longrightarrow -\infty$

x	$0 \longleftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \longleftarrow$	1000	100	10	2	1

Note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. The graph of f is shown in Figure 2.37.

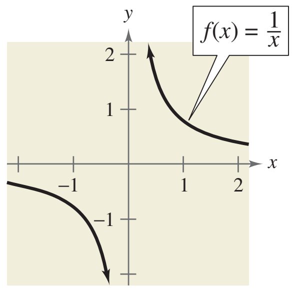


FIGURE 2.37

CHECKPoint Now try Exercise 5.

Vertical and Horizontal Asymptotes

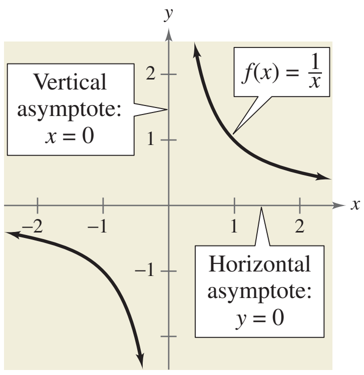


FIGURE 2.38

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \longrightarrow -\infty \text{ as } x \longrightarrow 0^-}_{f(x) \text{ decreases without bound as } x \text{ approaches 0 from the left.}} \qquad \underbrace{f(x) \longrightarrow \infty \text{ as } x \longrightarrow 0^+}_{f(x) \text{ increases without bound as } x \text{ approaches 0 from the right.}}$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.38. From this figure, you can see that the graph of f also has a **horizontal asymptote**—the line $y = 0$. This means that the values of $f(x) = \frac{1}{x}$ approach zero as x increases or decreases without bound.

$$\underbrace{f(x) \longrightarrow 0 \text{ as } x \longrightarrow -\infty}_{f(x) \text{ approaches 0 as } x \text{ decreases without bound.}} \qquad \underbrace{f(x) \longrightarrow 0 \text{ as } x \longrightarrow \infty}_{f(x) \text{ approaches 0 as } x \text{ increases without bound.}}$$

Definitions of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if

$$f(x) \longrightarrow \infty \quad \text{or} \quad f(x) \longrightarrow -\infty$$

as $x \longrightarrow a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f if

$$f(x) \longrightarrow b$$

as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.

Eventually (as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$), the distance between the horizontal asymptote and the points on the graph must approach zero. Figure 2.39 shows the vertical and horizontal asymptotes of the graphs of three rational functions.

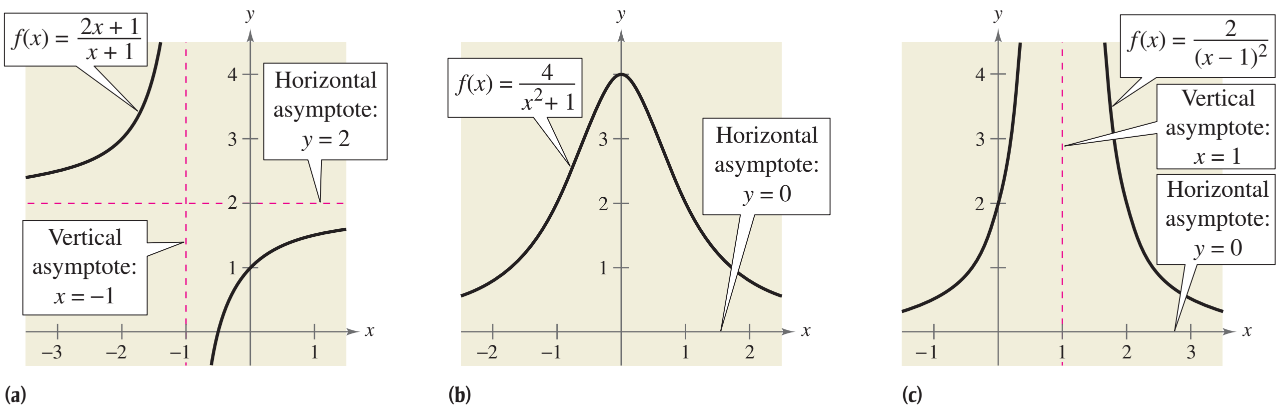


FIGURE 2.39

The graphs of $f(x) = \frac{1}{x}$ in Figure 2.38 and $f(x) = \frac{2x+1}{x+1}$ in Figure 2.39(a) are **hyperbolas.** You will study hyperbolas in Section 10.4.

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.

2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

- **a.** If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
- **b.** If $n = m$, the graph of f has the line $y = \frac{a_n}{b_m}$ (ratio of the leading coefficients) as a horizontal asymptote.
- **c.** If $n > m$, the graph of f has no horizontal asymptote.

Example 2 Finding Vertical and Horizontal Asymptotes

Find all vertical and horizontal asymptotes of the graph of each rational function.

a. $f(x) = \frac{2x^2}{x^2 - 1}$ **b.** $f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}$

Solution

a. For this rational function, the degree of the numerator is *equal to* the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$x^2 - 1 = 0$ Set denominator equal to zero.

$(x + 1)(x - 1) = 0$ Factor.

$x + 1 = 0 \Rightarrow x = -1$ Set 1st factor equal to 0.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0.

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes. The graph of the function is shown in Figure 2.40.

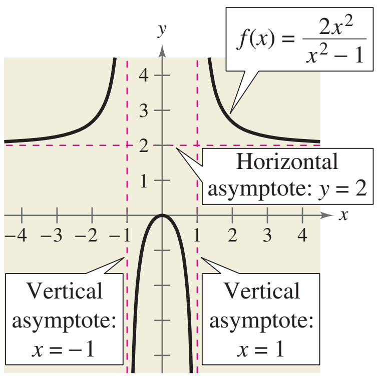


FIGURE 2.40

b. For this rational function, the degree of the numerator is *equal to* the degree of the denominator. The leading coefficient of both the numerator and denominator is 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote.

CHECK*Point* Now try Exercise 13.

Algebra Help

You can review the techniques for factoring in Appendix A.3.

Analyzing Graphs of Rational Functions

To sketch the graph of a rational function, use the following guidelines.

Study Tip

You may also want to test for symmetry when graphing rational functions, especially for simple rational functions. Recall from Section 1.6 that the graph of the reciprocal function

$$f(x) = \frac{1}{x}$$

is symmetric with respect to the origin.

Guidelines for Analyzing Graphs of Rational Functions

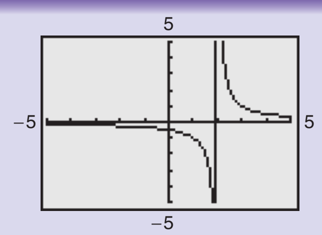
Let $f(x) = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials.

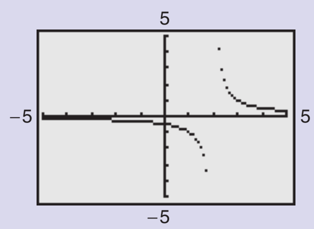
1. Simplify f, if possible.
2. Find and plot the y-intercept (if any) by evaluating $f(0)$.
3. Find the zeros of the numerator (if any) by solving the equation $N(x) = 0$. Then plot the corresponding x-intercepts.
4. Find the zeros of the denominator (if any) by solving the equation $D(x) = 0$. Then sketch the corresponding vertical asymptotes.
5. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.
6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

TECHNOLOGY

Some graphing utilities have difficulty graphing rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. For instance, the top screen on the right shows the graph of

$$f(x) = \frac{1}{x-2}.$$





Notice that the graph should consist of two unconnected portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the mode of the graphing utility to *dot mode*. The problem with this is that the graph is then represented as a collection of dots (as shown in the bottom screen on the right) rather than as a smooth curve.

The concept of *test intervals* from Section 2.2 can be extended to graphing of rational functions. To do this, use the fact that a rational function can change signs only at its zeros and its undefined values (the x-values for which its denominator is zero). Between two consecutive zeros of the numerator and the denominator, a rational function must be entirely positive or entirely negative. This means that when the zeros of the numerator and the denominator of a rational function are put in order, they divide the real number line into test intervals in which the function has no sign changes. A representative x-value is chosen to determine if the value of the rational function is positive (the graph lies above the x-axis) or negative (the graph lies below the x-axis).

Study Tip

You can use transformations to help you sketch graphs of rational functions. For instance, the graph of g in Example 3 is a vertical stretch and a right shift of the graph of $f(x) = 1/x$ because

$$g(x) = \frac{3}{x-2} = 3\left(\frac{1}{x-2}\right) = 3f(x-2).$$

Example 3 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x-2}$ and state its domain.

Solution

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: None, because $3 \neq 0$

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

Test interval	Representative x-value	Value of g	Sign	Point on graph
$(-\infty, 2)$	-4	$g(-4) = -0.5$	Negative	$(-4, -0.5)$
$(2, \infty)$	3	$g(3) = 3$	Positive	$(3, 3)$

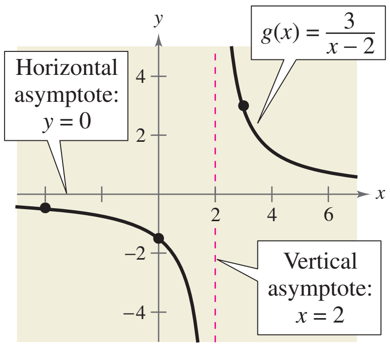


FIGURE 2.41

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.41. The domain of g is all real numbers x except $x = 2$.

CHECKPoint Now try Exercise 31.

Example 4 Sketching the Graph of a Rational Function

Sketch the graph of

$$f(x) = \frac{2x-1}{x}$$

and state its domain.

Solution

y-intercept: None, because $x = 0$ is not in the domain

x-intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$

Vertical asymptote: $x = 0$, zero of denominator

Horizontal asymptote: $y = 2$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

Test interval	Representative x-value	Value of f	Sign	Point on graph
$(-\infty, 0)$	-1	$f(-1) = 3$	Positive	$(-1, 3)$
$\left(0, \frac{1}{2}\right)$	$\frac{1}{4}$	$f\left(\frac{1}{4}\right) = -2$	Negative	$\left(\frac{1}{4}, -2\right)$
$\left(\frac{1}{2}, \infty\right)$	4	$f(4) = 1.75$	Positive	$(4, 1.75)$

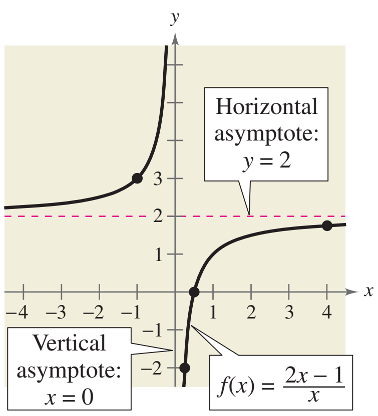


FIGURE 2.42

By plotting the intercepts, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.42. The domain of f is all real numbers x except $x = 0$.

CHECKPoint Now try Exercise 35.

Example 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = x/(x^2 - x - 2)$.

Solution

Factoring the denominator, you have $f(x) = \dfrac{x}{(x+1)(x-2)}$.

y-intercept: (0, 0), because $f(0) = 0$

x-intercept: (0, 0)

Vertical asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

Test interval	Representative *x*-value	Value of *f*	Sign	Point on graph
$(-\infty, -1)$	-3	$f(-3) = -0.3$	Negative	$(-3, -0.3)$
$(-1, 0)$	-0.5	$f(-0.5) = 0.4$	Positive	$(-0.5, 0.4)$
$(0, 2)$	1	$f(1) = -0.5$	Negative	$(1, -0.5)$
$(2, \infty)$	3	$f(3) = 0.75$	Positive	$(3, 0.75)$

The graph is shown in Figure 2.43.

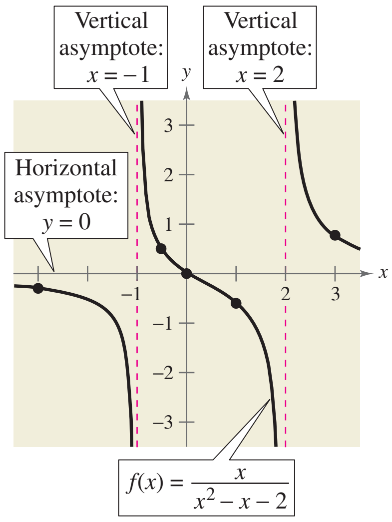


FIGURE 2.43

CHECKPoint Now try Exercise 39.

WARNING / CAUTION

If you are unsure of the shape of a portion of the graph of a rational function, plot some additional points. Also note that when the numerator and the denominator of a rational function have a common factor, the graph of the function has a *hole* at the zero of the common factor (see Example 6).

Example 6 A Rational Function with Common Factors

Sketch the graph of $f(x) = (x^2 - 9)/(x^2 - 2x - 3)$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+1)} = \frac{x+3}{x+1}, \quad x \neq 3.$$

y-intercept: (0, 3), because $f(0) = 3$

x-intercept: $(-3, 0)$, because $f(-3) = 0$

Vertical asymptote: $x = -1$, zero of (simplified) denominator

Horizontal asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

Test interval	Representative *x*-value	Value of *f*	Sign	Point on graph
$(-\infty, -3)$	-4	$f(-4) = 0.33$	Positive	$(-4, 0.33)$
$(-3, -1)$	-2	$f(-2) = -1$	Negative	$(-2, -1)$
$(-1, \infty)$	2	$f(2) = 1.67$	Positive	$(2, 1.67)$

The graph is shown in Figure 2.44. Notice that there is a hole in the graph at $x = 3$, because the function is not defined when $x = 3$.

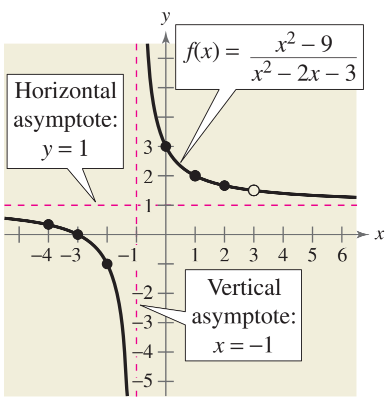


FIGURE 2.44 Hole at $x = 3$

CHECKPoint Now try Exercise 45.

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.45. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you obtain

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\text{Slant asymptote } (y = x - 2)} + \frac{2}{x + 1}.$$

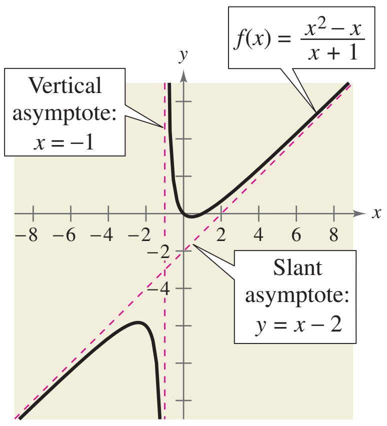


FIGURE 2.45

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 2.45.

Example 7 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

Factoring the numerator as $(x - 2)(x + 1)$ allows you to recognize the x-intercepts. Using long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

allows you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-intercept: $(0, 2)$, because $f(0) = 2$

x-intercepts: $(-1, 0)$ and $(2, 0)$

Vertical asymptote: $x = 1$, zero of denominator

Slant asymptote: $y = x$

Additional points:

Test interval	Representative x-value	Value of f	Sign	Point on graph
$(-\infty, -1)$	-2	$f(-2) = -1.33$	Negative	$(-2, -1.33)$
$(-1, 1)$	0.5	$f(0.5) = 4.5$	Positive	$(0.5, 4.5)$
$(1, 2)$	1.5	$f(1.5) = -2.5$	Negative	$(1.5, -2.5)$
$(2, \infty)$	3	$f(3) = 2$	Positive	$(3, 2)$

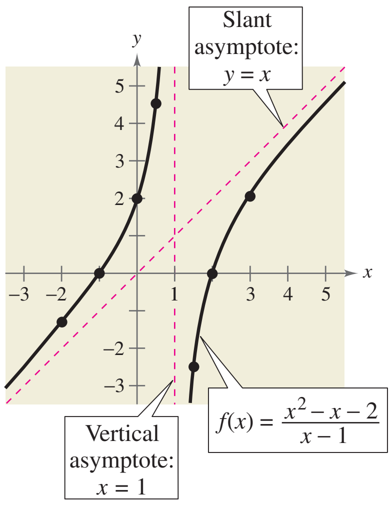


FIGURE 2.46

The graph is shown in Figure 2.46.

CHECKPoint Now try Exercise 65.

Applications

There are many examples of asymptotic behavior in real life. For instance, Example 8 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

Example 8 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing p% of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}$$

for $0 \le p < 100$. You are a member of a state legislature considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Algebraic Solution

Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \approx \$453{,}333. \qquad \text{Evaluate } C \text{ when } p = 85.$$

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000(90)}{100 - 90} = \$720{,}000. \qquad \text{Evaluate } C \text{ when } p = 90.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

Graphical Solution

Use a graphing utility to graph the function

$$y_1 = \frac{80{,}000}{100 - x}$$

using a viewing window similar to that shown in Figure 2.47. Note that the graph has a vertical asymptote at $x = 100$. Then use the *trace* or *value* feature to approximate the values of y_1 when $x = 85$ and $x = 90$. You should obtain the following values.

When $x = 85$, $y_1 \approx 453{,}333$.

When $x = 90$, $y_1 = 720{,}000$.

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667.$$

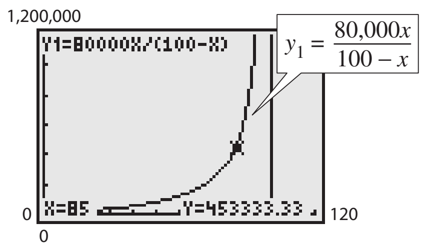


FIGURE 2.47

CHECKPoint Now try Exercise 77.

Example 9 Finding a Minimum Area

A rectangular page is designed to contain 48 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be so that the least amount of paper is used?

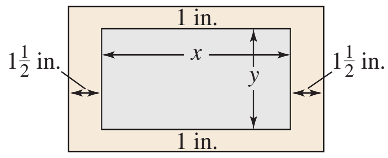


FIGURE 2.48

Graphical Solution

Let A be the area to be minimized. From Figure 2.48, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right)$$

$$= \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 2.49. Because x represents the width of the printed area, you need consider only the portion of the graph for which x is positive. Using a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches} \quad \text{by} \quad y + 2 \approx 7.6 \text{ inches.}$$

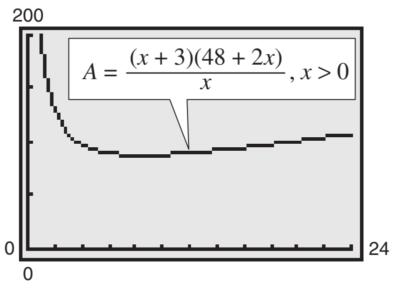


FIGURE 2.49

CHECK *Point* Now try Exercise 81.

Numerical Solution

Let A be the area to be minimized. From Figure 2.48, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

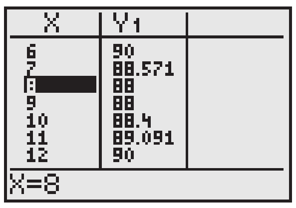
$$A = (x + 3)\left(\frac{48}{x} + 2\right)$$

$$= \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function

$$y_1 = \frac{(x + 3)(48 + 2x)}{x}$$

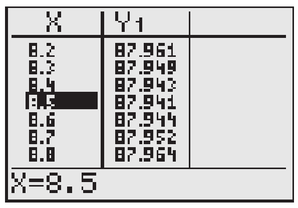
beginning at $x = 1$. From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.50. To approximate the minimum value of y_1 to one decimal place, change the table so that it starts at $x = 8$ and increases by 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.51. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.



X	Y1
6	90
7	88.571
8	88
9	88
10	88.4
11	89.091
12	90

X=8

FIGURE 2.50



X	Y1
8.2	87.961
8.3	87.949
8.4	87.943
8.5	87.941
8.6	87.944
8.7	87.952
8.8	87.964

X=8.5

FIGURE 2.51

If you go on to take a course in calculus, you will learn an analytic technique for finding the exact value of x that produces a minimum area. In this case, that value is $x = 6\sqrt{2} \approx 8.485$.

2.6 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called ________ ________.
2. If $f(x) \to \pm\infty$ as $x \to a$ from the left or the right, then $x = a$ is a ________ ________ of the graph of f.
3. If $f(x) \to b$ as $x \to \pm\infty$, then $y = b$ is a ________ ________ of the graph of f.
4. For the rational function given by $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a ________ (or oblique) ________.

SKILLS AND APPLICATIONS

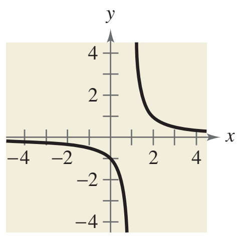
In Exercises 5–8, (a) complete each table for the function, (b) determine the vertical and horizontal asymptotes of the graph of the function, and (c) find the domain of the function.

x	$f(x)$
0.5	
0.9	
0.99	
0.999	

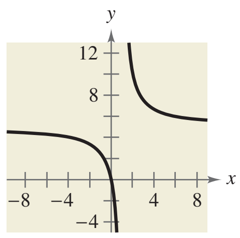
x	$f(x)$
1.5	
1.1	
1.01	
1.001	

x	$f(x)$
5	
10	
100	
1000	

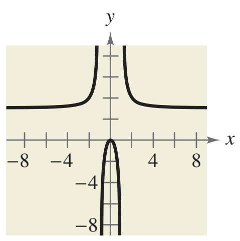
5. $f(x) = \dfrac{1}{x-1}$



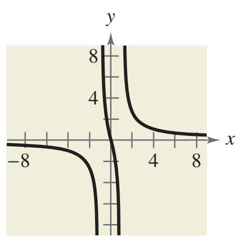
6. $f(x) = \dfrac{5x}{x-1}$



7. $f(x) = \dfrac{3x^2}{x^2-1}$



8. $f(x) = \dfrac{4x}{x^2-1}$



In Exercises 9–16, find the domain of the function and identify any vertical and horizontal asymptotes.

9. $f(x) = \dfrac{4}{x^2}$

10. $f(x) = \dfrac{4}{(x-2)^3}$

11. $f(x) = \dfrac{5+x}{5-x}$

12. $f(x) = \dfrac{3-7x}{3+2x}$

13. $f(x) = \dfrac{x^3}{x^2-1}$

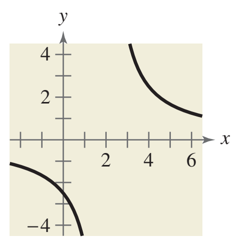
14. $f(x) = \dfrac{4x^2}{x+2}$

15. $f(x) = \dfrac{3x^2+1}{x^2+x+9}$

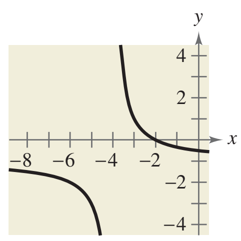
16. $f(x) = \dfrac{3x^2+x-5}{x^2+1}$

In Exercises 17–20, match the rational function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

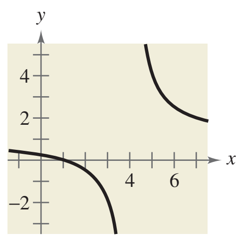
(a)



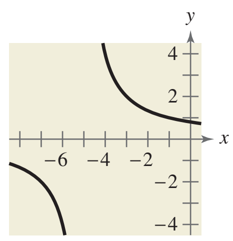
(b)



(c)



(d)



17. $f(x) = \dfrac{4}{x+5}$

18. $f(x) = \dfrac{5}{x-2}$

19. $f(x) = \dfrac{x-1}{x-4}$

20. $f(x) = -\dfrac{x+2}{x+4}$

In Exercises 21–24, find the zeros (if any) of the rational function.

21. $g(x) = \dfrac{x^2-9}{x+3}$

22. $h(x) = 4 + \dfrac{10}{x^2+5}$

23. $f(x) = 1 - \dfrac{2}{x-7}$

24. $g(x) = \dfrac{x^3-8}{x^2+1}$

In Exercises 25–30, find the domain of the function and identify any vertical and horizontal asymptotes.

25. $f(x) = \dfrac{x - 4}{x^2 - 16}$

26. $f(x) = \dfrac{x + 1}{x^2 - 1}$

27. $f(x) = \dfrac{x^2 - 25}{x^2 - 4x - 5}$

28. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}$

29. $f(x) = \dfrac{x^2 - 3x - 4}{2x^2 + x - 1}$

30. $f(x) = \dfrac{6x^2 - 11x + 3}{6x^2 - 7x - 3}$

In Exercises 31–50, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical and horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

31. $f(x) = \dfrac{1}{x + 2}$

32. $f(x) = \dfrac{1}{x - 3}$

33. $h(x) = \dfrac{-1}{x + 4}$

34. $g(x) = \dfrac{1}{6 - x}$

35. $C(x) = \dfrac{7 + 2x}{2 + x}$

36. $P(x) = \dfrac{1 - 3x}{1 - x}$

37. $f(x) = \dfrac{x^2}{x^2 + 9}$

38. $f(t) = \dfrac{1 - 2t}{t}$

39. $g(s) = \dfrac{4s}{s^2 + 4}$

40. $f(x) = -\dfrac{1}{(x - 2)^2}$

41. $h(x) = \dfrac{x^2 - 5x + 4}{x^2 - 4}$

42. $g(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$

43. $f(x) = \dfrac{2x^2 - 5x - 3}{x^3 - 2x^2 - x + 2}$

44. $f(x) = \dfrac{x^2 - x - 2}{x^3 - 2x^2 - 5x + 6}$

45. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$

46. $f(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$

47. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - x - 6}$

48. $f(x) = \dfrac{3x^2 - 8x + 4}{2x^2 - 3x - 2}$

49. $f(t) = \dfrac{t^2 - 1}{t - 1}$

50. $f(x) = \dfrac{x^2 - 36}{x + 6}$

ANALYTICAL, NUMERICAL, AND GRAPHICAL ANALYSIS In Exercises 51–54, do the following.

(a) Determine the domains of f and g.

(b) Simplify f and find any vertical asymptotes of the graph of f.

(c) Compare the functions by completing the table.

(d) Use a graphing utility to graph f and g in the same viewing window.

(e) Explain why the graphing utility may not show the difference in the domains of f and g.

51. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad g(x) = x - 1$

x	-3	-2	-1.5	-1	-0.5	0	1
$f(x)$							
$g(x)$							

52. $f(x) = \dfrac{x^2(x - 2)}{x^2 - 2x}, \quad g(x) = x$

x	-1	0	1	1.5	2	2.5	3
$f(x)$							
$g(x)$							

53. $f(x) = \dfrac{x - 2}{x^2 - 2x}, \quad g(x) = \dfrac{1}{x}$

x	-0.5	0	0.5	1	1.5	2	3
$f(x)$							
$g(x)$							

54. $f(x) = \dfrac{2x - 6}{x^2 - 7x + 12}, \quad g(x) = \dfrac{2}{x - 4}$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

In Exercises 55–68, (a) state the domain of the function, (b) identify all intercepts, (c) identify any vertical and slant asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

55. $h(x) = \dfrac{x^2 - 9}{x}$

56. $g(x) = \dfrac{x^2 + 5}{x}$

57. $f(x) = \dfrac{2x^2 + 1}{x}$

58. $f(x) = \dfrac{1 - x^2}{x}$

59. $g(x) = \dfrac{x^2 + 1}{x}$

60. $h(x) = \dfrac{x^2}{x - 1}$

61. $f(t) = -\dfrac{t^2 + 1}{t + 5}$

62. $f(x) = \dfrac{x^2}{3x + 1}$

63. $f(x) = \dfrac{x^3}{x^2 - 4}$

64. $g(x) = \dfrac{x^3}{2x^2 - 8}$

65. $f(x) = \dfrac{x^2 - x + 1}{x - 1}$

66. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

67. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

68. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

In Exercises 69–72, use a graphing utility to graph the rational function. Give the domain of the function and identify any asymptotes. Then zoom out sufficiently far so that the graph appears as a line. Identify the line.

69. $f(x) = \dfrac{x^2 + 5x + 8}{x + 3}$

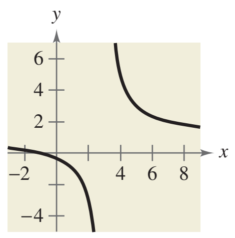
70. $f(x) = \dfrac{2x^2 + x}{x + 1}$

71. $g(x) = \dfrac{1 + 3x^2 - x^3}{x^2}$

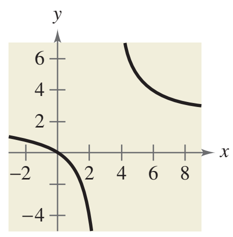
72. $h(x) = \dfrac{12 - 2x - x^2}{2(4 + x)}$

GRAPHICAL REASONING In Exercises 73–76, (a) use the graph to determine any x-intercepts of the graph of the rational function and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).

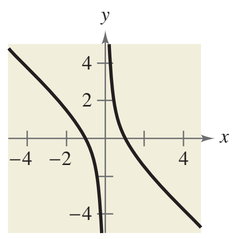
73. $y = \dfrac{x + 1}{x - 3}$



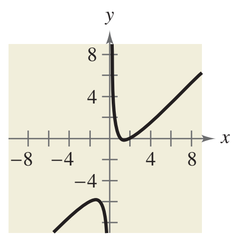
74. $y = \dfrac{2x}{x - 3}$



75. $y = \dfrac{1}{x} - x$



76. $y = x - 3 + \dfrac{2}{x}$



77. POLLUTION The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of removing 10%, 40%, and 75% of the pollutants.

(c) According to this model, would it be possible to remove 100% of the pollutants? Explain.

78. RECYCLING In a pilot project, a rural township is given recycling bins for separating and storing recyclable products. The cost C (in dollars) of supplying bins to $p\%$ of the population is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

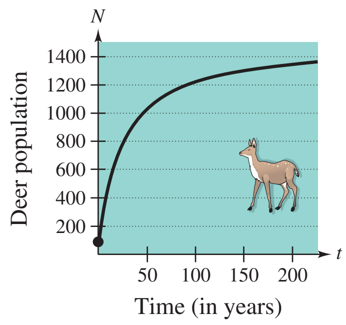
(b) Find the costs of supplying bins to 15%, 50%, and 90% of the population.

(c) According to this model, would it be possible to supply bins to 100% of the residents? Explain.

79. POPULATION GROWTH The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is modeled by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \ge 0$$

where t is the time in years (see figure).



(a) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(b) What is the limiting size of the herd as time increases?

80. CONCENTRATION OF A MIXTURE A 1000-liter tank contains 50 liters of a 25% brine solution. You add x liters of a 75% brine solution to the tank.

(a) Show that the concentration C, the proportion of brine to total solution, in the final mixture is

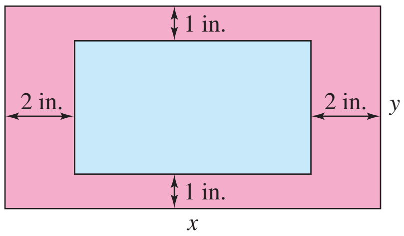
$$C = \frac{3x + 50}{4(x + 50)}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the concentration function.

(d) As the tank is filled, what happens to the rate at which the concentration of brine is increasing? What percent does the concentration of brine appear to approach?

81. PAGE DESIGN A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are 1 inch deep, and the margins on each side are 2 inches wide (see figure).



(a) Write a function for the total area A of the page in terms of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size for which the least amount of paper will be used. Verify your answer numerically using the *table* feature of the graphing utility.

82. PAGE DESIGN A rectangular page is designed to contain 64 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be so that the least amount of paper is used?

83. AVERAGE SPEED A driver averaged 50 miles per hour on the round trip between Akron, Ohio, and Columbus, Ohio, 100 miles away. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = \dfrac{25x}{x - 25}$.

(b) Determine the vertical and horizontal asymptotes of the graph of the function.

(c) Use a graphing utility to graph the function.

(d) Complete the table.

x	30	35	40	45	50	55	60
y							

(e) Are the results in the table what you expected? Explain.

(f) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

EXPLORATION

84. WRITING Is every rational function a polynomial function? Is every polynomial function a rational function? Explain.

TRUE OR FALSE? In Exercises 85–87, determine whether the statement is true or false. Justify your answer.

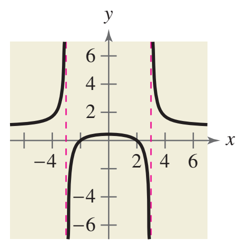
85. A polynomial can have infinitely many vertical asymptotes.

86. The graph of a rational function can never cross one of its asymptotes.

87. The graph of a function can have a vertical asymptote, a horizontal asymptote, and a slant asymptote.

LIBRARY OF PARENT FUNCTIONS In Exercises 88 and 89, identify the rational function represented by the graph.

88.



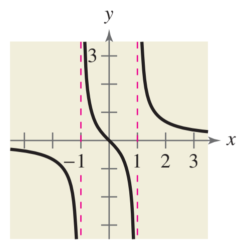
(a) $f(x) = \dfrac{x^2 - 9}{x^2 - 4}$

(b) $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$

(c) $f(x) = \dfrac{x - 4}{x^2 - 9}$

(d) $f(x) = \dfrac{x - 9}{x^2 - 4}$

89.



(a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(b) $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

(c) $f(x) = \dfrac{x}{x^2 - 1}$

(d) $f(x) = \dfrac{x}{x^2 + 1}$

90. CAPSTONE Write a rational function f that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$
Zero: $x = 1$

(b) Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 0$
Zero: $x = 2$

(c) Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: $y = 2$
Zeros: $x = 3, x = -3,$

(d) Vertical asymptotes: $x = -1, x = 2$
Horizontal asymptote: $y = -2$
Zeros: $x = -2, x = 3$

PROJECT: DEPARTMENT OF DEFENSE To work an extended application analyzing the total numbers of the Department of Defense personnel from 1980 through 2007, visit this text's website at *academic.cengage.com*. (Data Source: U.S. Department of Defense)

2.7 NONLINEAR INEQUALITIES

What you should learn

- Solve polynomial inequalities.
- Solve rational inequalities.
- Use inequalities to model and solve real-life problems.

Why you should learn it

Inequalities can be used to model and solve real-life problems. For instance, in Exercise 77 on page 202, a polynomial inequality is used to model school enrollment in the United States.

Ellen Senisi/The Image Works

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, you can use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **key numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality. For instance, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$. These zeros divide the real number line into three test intervals:

$$(-\infty, -1), \quad (-1, 3), \quad \text{and} \quad (3, \infty). \qquad \text{(See Figure 2.52.)}$$

So, to solve the inequality $x^2 - 2x - 3 < 0$, you need only test one value from each of these test intervals to determine whether the value satisfies the original inequality. If so, you can conclude that the interval is a solution of the inequality.

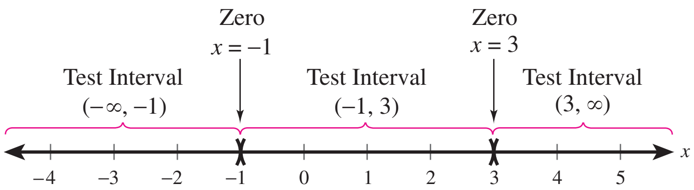


FIGURE 2.52 Three test intervals for $x^2 - 2x - 3$

You can use the same basic approach to determine the test intervals for any polynomial.

Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order (from smallest to largest). These zeros are the key numbers of the polynomial.
2. Use the key numbers of the polynomial to determine its test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, the polynomial will have negative values for every x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for every x-value in the interval.

Algebra Help

You can review the techniques for factoring polynomials in Appendix A.3.

Example 1 Solving a Polynomial Inequality

Solve $x^2 - x - 6 < 0$.

Solution

By factoring the polynomial as

$$x^2 - x - 6 = (x + 2)(x - 3)$$

you can see that the key numbers are $x = -2$ and $x = 3$. So, the polynomial's test intervals are

$$(-\infty, -2), \quad (-2, 3), \quad \text{and} \quad (3, \infty). \qquad \text{Test intervals}$$

In each test interval, choose a representative x-value and evaluate the polynomial.

Test Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 4$	$(4)^2 - (4) - 6 = 6$	Positive

From this you can conclude that the inequality is satisfied for all x-values in $(-2, 3)$. This implies that the solution of the inequality $x^2 - x - 6 < 0$ is the interval $(-2, 3)$, as shown in Figure 2.53. Note that the original inequality contains a "less than" symbol. This means that the solution set does not contain the endpoints of the test interval $(-2, 3)$.

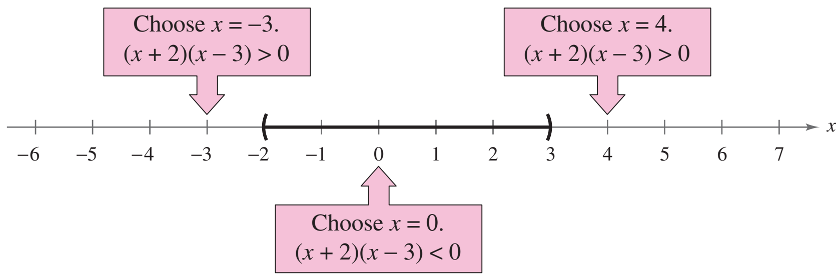


FIGURE 2.53

CHECKPoint Now try Exercise 21.

As with linear inequalities, you can check the reasonableness of a solution by substituting x-values into the original inequality. For instance, to check the solution found in Example 1, try substituting several x-values from the interval $(-2, 3)$ into the inequality

$$x^2 - x - 6 < 0.$$

Regardless of which x-values you choose, the inequality should be satisfied.

You can also use a graph to check the result of Example 1. Sketch the graph of $y = x^2 - x - 6$, as shown in Figure 2.54. Notice that the graph is below the x-axis on the interval $(-2, 3)$.

In Example 1, the polynomial inequality was given in general form (with the polynomial on one side and zero on the other). Whenever this is not the case, you should begin the solution process by writing the inequality in general form.

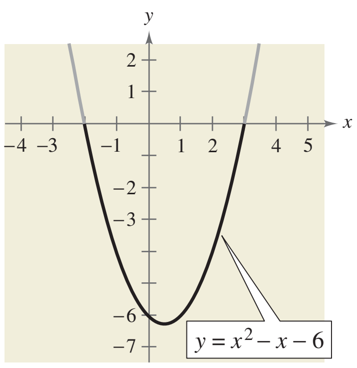


FIGURE 2.54

Example 2 Solving a Polynomial Inequality

Solve $2x^3 - 3x^2 - 32x > -48$.

Solution

$$2x^3 - 3x^2 - 32x + 48 > 0 \qquad \text{Write in general form.}$$

$$(x - 4)(x + 4)(2x - 3) > 0 \qquad \text{Factor.}$$

The key numbers are $x = -4, x = \frac{3}{2}$, and $x = 4$, and the test intervals are $(-\infty, -4)$, $\left(-4, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 4\right)$, and $(4, \infty)$.

Test Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -4)$	$x = -5$	$2(-5)^3 - 3(-5)^2 - 32(-5) + 48$	Negative
$\left(-4, \frac{3}{2}\right)$	$x = 0$	$2(0)^3 - 3(0)^2 - 32(0) + 48$	Positive
$\left(\frac{3}{2}, 4\right)$	$x = 2$	$2(2)^3 - 3(2)^2 - 32(2) + 48$	Negative
$(4, \infty)$	$x = 5$	$2(5)^3 - 3(5)^2 - 32(5) + 48$	Positive

From this you can conclude that the inequality is satisfied on the open intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$. So, the solution set is $\left(-4, \frac{3}{2}\right) \cup (4, \infty)$, as shown in Figure 2.55.

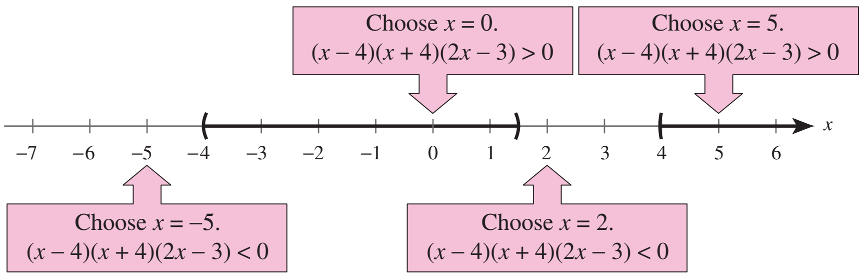


FIGURE 2.55

CHECKPoint Now try Exercise 27.

Example 3 Solving a Polynomial Inequality

Solve $4x^2 - 5x > 6$.

Algebraic Solution

$$4x^2 - 5x - 6 > 0 \qquad \text{Write in general form.}$$

$$(x - 2)(4x + 3) > 0 \qquad \text{Factor.}$$

Key Numbers: $x = -\frac{3}{4}, \quad x = 2$

Test Intervals: $\left(-\infty, -\frac{3}{4}\right), \left(-\frac{3}{4}, 2\right), (2, \infty)$

Test: Is $(x - 2)(4x + 3) > 0$?

After testing these intervals, you can see that the polynomial $4x^2 - 5x - 6$ is positive on the open intervals $\left(-\infty, -\frac{3}{4}\right)$ and $(2, \infty)$. So, the solution set of the inequality is $\left(-\infty, -\frac{3}{4}\right) \cup (2, \infty)$.

CHECKPoint Now try Exercise 23.

Graphical Solution

First write the polynomial inequality $4x^2 - 5x > 6$ as $4x^2 - 5x - 6 > 0$. Then use a graphing utility to graph $y = 4x^2 - 5x - 6$. In Figure 2.56, you can see that the graph is *above* the x-axis when x is less than $-\frac{3}{4}$ *or* when x is greater than 2. So, you can graphically approximate the solution set to be $\left(-\infty, -\frac{3}{4}\right) \cup (2, \infty)$.

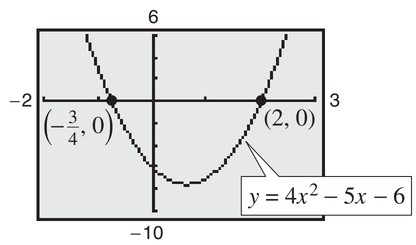


FIGURE 2.56

Study Tip

You may find it easier to determine the sign of a polynomial from its *factored* form. For instance, in Example 3, if the test value $x = 1$ is substituted into the factored form

$$(x - 2)(4x + 3)$$

you can see that the sign pattern of the factors is

$$(\ -\ )(\ +\ )$$

which yields a negative result. Try using the factored forms of the polynomials to determine the signs of the polynomials in the test intervals of the other examples in this section.

When solving a polynomial inequality, be sure you have accounted for the particular type of inequality symbol given in the inequality. For instance, in Example 3, note that the original inequality contained a "greater than" symbol and the solution consisted of two open intervals. If the original inequality had been

$$4x^2 - 5x \geq 6$$

the solution would have consisted of the intervals $\left(-\infty, -\frac{3}{4}\right]$ and $[2, \infty)$.

Each of the polynomial inequalities in Examples 1, 2, and 3 has a solution set that consists of a single interval or the union of two intervals. When solving the exercises for this section, watch for unusual solution sets, as illustrated in Example 4.

Example 4 Unusual Solution Sets

a. The solution set of the following inequality consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x.

$$x^2 + 2x + 4 > 0$$

b. The solution set of the following inequality consists of the single real number $\{-1\}$, because the quadratic $x^2 + 2x + 1$ has only one key number, $x = -1$, and it is the only value that satisfies the inequality.

$$x^2 + 2x + 1 \leq 0$$

c. The solution set of the following inequality is empty. In other words, the quadratic $x^2 + 3x + 5$ is not less than zero for any value of x.

$$x^2 + 3x + 5 < 0$$

d. The solution set of the following inequality consists of all real numbers except $x = 2$. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$.

$$x^2 - 4x + 4 > 0$$

CHECKPoint Now try Exercise 29.

Rational Inequalities

The concepts of key numbers and test intervals can be extended to rational inequalities. To do this, use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the *key numbers* of a rational inequality. When solving a rational inequality, begin by writing the inequality in general form with the rational expression on the left and zero on the right.

Study Tip

In Example 5, if you write 3 as $\frac{3}{1}$, you should be able to see that the LCD (least common denominator) is $(x - 5)(1) = x - 5$. So, you can rewrite the general form as

$$\frac{2x - 7}{x - 5} - \frac{3(x - 5)}{x - 5} \le 0,$$

which simplifies as shown.

Example 5 Solving a Rational Inequality

Solve $\dfrac{2x - 7}{x - 5} \le 3$.

Solution

$\dfrac{2x - 7}{x - 5} \le 3$ Write original inequality.

$\dfrac{2x - 7}{x - 5} - 3 \le 0$ Write in general form.

$\dfrac{2x - 7 - 3x + 15}{x - 5} \le 0$ Find the LCD and subtract fractions.

$\dfrac{-x + 8}{x - 5} \le 0$ Simplify.

Key numbers: $x = 5, x = 8$ Zeros and undefined values of rational expression

Test intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\dfrac{-x + 8}{x - 5} \le 0$?

After testing these intervals, as shown in Figure 2.57, you can see that the inequality is satisfied on the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $\dfrac{-x + 8}{x - 5} = 0$ when $x = 8$, you can conclude that the solution set consists of all real numbers in the intervals $(-\infty, 5) \cup [8, \infty)$. (Be sure to use a closed interval to indicate that x can equal 8.)

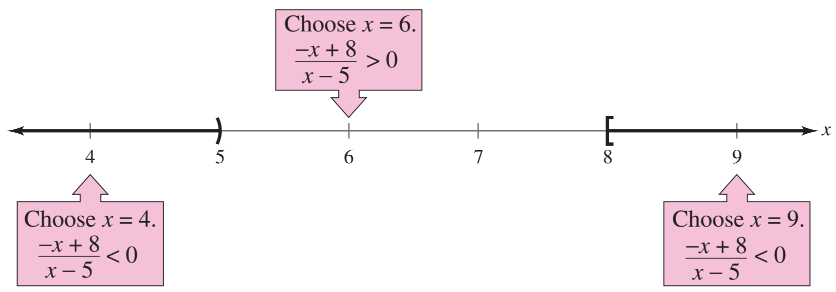


FIGURE 2.57

CHECKPoint Now try Exercise 45.

Applications

One common application of inequalities comes from business and involves profit, revenue, and cost. The formula that relates these three quantities is

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$P = R - C.$$

Example 6 Increasing the Profit for a Product

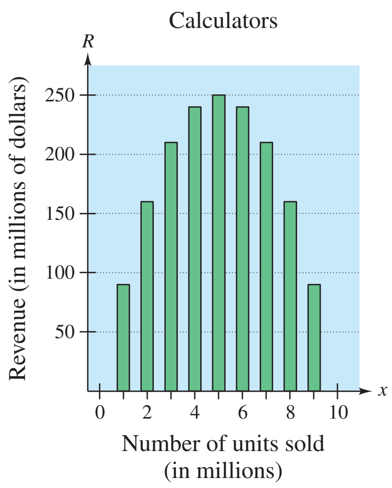


FIGURE 2.58

The marketing department of a calculator manufacturer has determined that the demand for a new model of calculator is

$$p = 100 - 0.00001x, \quad 0 \le x \le 10{,}000{,}000 \qquad \text{Demand equation}$$

where p is the price per calculator (in dollars) and x represents the number of calculators sold. (If this model is accurate, no one would be willing to pay \$100 for the calculator. At the other extreme, the company couldn't sell more than 10 million calculators.) The revenue for selling x calculators is

$$R = xp = x(100 - 0.00001x) \qquad \text{Revenue equation}$$

as shown in Figure 2.58. The total cost of producing x calculators is \$10 per calculator plus a development cost of \$2,500,000. So, the total cost is

$$C = 10x + 2{,}500{,}000. \qquad \text{Cost equation}$$

What price should the company charge per calculator to obtain a profit of at least \$190,000,000?

Solution

Verbal Model: $\text{Profit} = \text{Revenue} - \text{Cost}$

Equation: $P = R - C$

$$P = 100x - 0.00001x^2 - (10x + 2{,}500{,}000)$$

$$P = -0.00001x^2 + 90x - 2{,}500{,}000$$

To answer the question, solve the inequality

$$P \ge 190{,}000{,}000$$

$$-0.00001x^2 + 90x - 2{,}500{,}000 \ge 190{,}000{,}000.$$

When you write the inequality in general form, find the key numbers and the test intervals, and then test a value in each test interval, you can find the solution to be

$$3{,}500{,}000 \le x \le 5{,}500{,}000$$

as shown in Figure 2.59. Substituting the x-values in the original price equation shows that prices of

$$\$45.00 \le p \le \$65.00$$

will yield a profit of at least \$190,000,000.

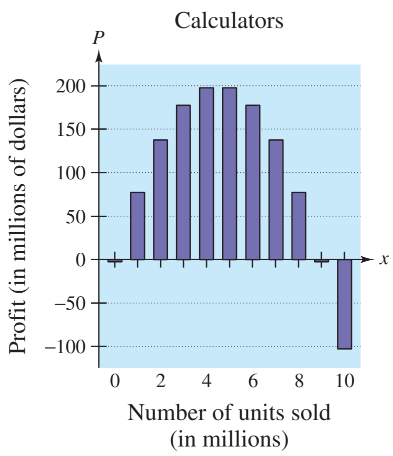


FIGURE 2.59

CHECKPoint Now try Exercise 75.

Another common application of inequalities is finding the domain of an expression that involves a square root, as shown in Example 7.

Example 7 Finding the Domain of an Expression

Find the domain of $\sqrt{64 - 4x^2}$.

Algebraic Solution

Remember that the domain of an expression is the set of all x-values for which the expression is defined. Because $\sqrt{64 - 4x^2}$ is defined (has real values) only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$64 - 4x^2 \geq 0$ Write in general form.

$16 - x^2 \geq 0$ Divide each side by 4.

$(4 - x)(4 + x) \geq 0$ Write in factored form.

So, the inequality has two key numbers: $x = -4$ and $x = 4$. You can use these two numbers to test the inequality as follows.

Key numbers: $x = -4, x = 4$

Test intervals: $(-\infty, -4), (-4, 4), (4, \infty)$

Test: For what values of x is $\sqrt{64 - 4x^2} \geq 0$?

A test shows that the inequality is satisfied in the *closed interval* $[-4, 4]$. So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$.

CHECK*Point* Now try Exercise 59.

Graphical Solution

Begin by sketching the graph of the equation $y = \sqrt{64 - 4x^2}$, as shown in Figure 2.60. From the graph, you can determine that the x-values extend from -4 to 4 (including -4 and 4). So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$.

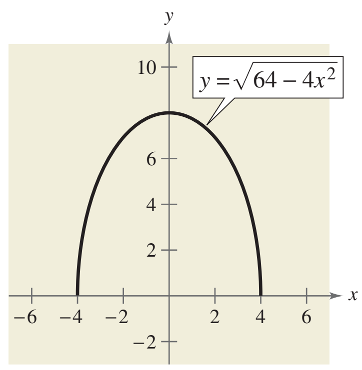


FIGURE 2.60

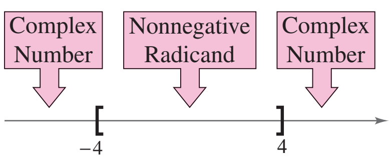


FIGURE 2.61

To analyze a test interval, choose a representative x-value in the interval and evaluate the expression at that value. For instance, in Example 7, if you substitute any number from the interval $[-4, 4]$ into the expression $\sqrt{64 - 4x^2}$, you will obtain a nonnegative number under the radical symbol that simplifies to a real number. If you substitute any number from the intervals $(-\infty, -4)$ and $(4, \infty)$, you will obtain a complex number. It might be helpful to draw a visual representation of the intervals, as shown in Figure 2.61.

Classroom Discussion

Profit Analysis Consider the relationship

$$P = R - C$$

described on page 199. Write a paragraph discussing why it might be beneficial to solve $P < 0$ if you owned a business. Use the situation described in Example 6 to illustrate your reasoning.

2.7 EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

VOCABULARY: Fill in the blanks.

1. Between two consecutive zeros, a polynomial must be entirely ________ or entirely ________.
2. To solve a polynomial inequality, find the ________ numbers of the polynomial, and use these numbers to create ________ ________ for the inequality.
3. The key numbers of a rational expression are its ________ and its ________ ________.
4. The formula that relates cost, revenue, and profit is ________.

SKILLS AND APPLICATIONS

In Exercises 5–8, determine whether each value of x is a solution of the inequality.

Inequality	*Values*	
5. $x^2 - 3 < 0$	(a) $x = 3$	(b) $x = 0$
	(c) $x = \frac{3}{2}$	(d) $x = -5$
6. $x^2 - x - 12 \geq 0$	(a) $x = 5$	(b) $x = 0$
	(c) $x = -4$	(d) $x = -3$
7. $\dfrac{x+2}{x-4} \geq 3$	(a) $x = 5$	(b) $x = 4$
	(c) $x = -\frac{9}{2}$	(d) $x = \frac{9}{2}$
8. $\dfrac{3x^2}{x^2+4} < 1$	(a) $x = -2$	(b) $x = -1$
	(c) $x = 0$	(d) $x = 3$

In Exercises 9–12, find the key numbers of the expression.

9. $3x^2 - x - 2$

10. $9x^3 - 25x^2$

11. $\dfrac{1}{x-5} + 1$

12. $\dfrac{x}{x+2} - \dfrac{2}{x-1}$

In Exercises 13–30, solve the inequality and graph the solution on the real number line.

13. $x^2 < 9$

14. $x^2 \leq 16$

15. $(x + 2)^2 \leq 25$

16. $(x - 3)^2 \geq 1$

17. $x^2 + 4x + 4 \geq 9$

18. $x^2 - 6x + 9 < 16$

19. $x^2 + x < 6$

20. $x^2 + 2x > 3$

21. $x^2 + 2x - 3 < 0$

22. $x^2 > 2x + 8$

23. $3x^2 - 11x > 20$

24. $-2x^2 + 6x + 15 \leq 0$

25. $x^2 - 3x - 18 > 0$

26. $x^3 + 2x^2 - 4x - 8 \leq 0$

27. $x^3 - 3x^2 - x > -3$

28. $2x^3 + 13x^2 - 8x - 46 \geq 6$

29. $4x^2 - 4x + 1 \leq 0$

30. $x^2 + 3x + 8 > 0$

In Exercises 31–36, solve the inequality and write the solution set in interval notation.

31. $4x^3 - 6x^2 < 0$

32. $4x^3 - 12x^2 > 0$

33. $x^3 - 4x \geq 0$

34. $2x^3 - x^4 \leq 0$

35. $(x - 1)^2(x + 2)^3 \geq 0$

36. $x^4(x - 3) \leq 0$

GRAPHICAL ANALYSIS In Exercises 37–40, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.

Equation	*Inequalities*	
37. $y = -x^2 + 2x + 3$	(a) $y \leq 0$	(b) $y \geq 3$
38. $y = \frac{1}{2}x^2 - 2x + 1$	(a) $y \leq 0$	(b) $y \geq 7$
39. $y = \frac{1}{8}x^3 - \frac{1}{2}x$	(a) $y \geq 0$	(b) $y \leq 6$
40. $y = x^3 - x^2 - 16x + 16$	(a) $y \leq 0$	(b) $y \geq 36$

In Exercises 41–54, solve the inequality and graph the solution on the real number line.

41. $\dfrac{4x-1}{x} > 0$

42. $\dfrac{x^2-1}{x} < 0$

43. $\dfrac{3x-5}{x-5} \geq 0$

44. $\dfrac{5+7x}{1+2x} \leq 4$

45. $\dfrac{x+6}{x+1} - 2 < 0$

46. $\dfrac{x+12}{x+2} - 3 \geq 0$

47. $\dfrac{2}{x+5} > \dfrac{1}{x-3}$

48. $\dfrac{5}{x-6} > \dfrac{3}{x+2}$

49. $\dfrac{1}{x-3} \leq \dfrac{9}{4x+3}$

50. $\dfrac{1}{x} \geq \dfrac{1}{x+3}$

51. $\dfrac{x^2+2x}{x^2-9} \leq 0$

52. $\dfrac{x^2+x-6}{x} \geq 0$

53. $\dfrac{3}{x-1} + \dfrac{2x}{x+1} > -1$

54. $\dfrac{3x}{x-1} \leq \dfrac{x}{x+4} + 3$

GRAPHICAL ANALYSIS In Exercises 55–58, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.

Equation	*Inequalities*	
55. $y = \dfrac{3x}{x - 2}$	(a) $y \le 0$	(b) $y \ge 6$
56. $y = \dfrac{2(x - 2)}{x + 1}$	(a) $y \le 0$	(b) $y \ge 8$
57. $y = \dfrac{2x^2}{x^2 + 4}$	(a) $y \ge 1$	(b) $y \le 2$
58. $y = \dfrac{5x}{x^2 + 4}$	(a) $y \ge 1$	(b) $y \le 0$

In Exercises 59–64, find the domain of x in the expression. Use a graphing utility to verify your result.

59. $\sqrt{4 - x^2}$ **60.** $\sqrt{x^2 - 4}$

61. $\sqrt{x^2 - 9x + 20}$ **62.** $\sqrt{81 - 4x^2}$

63. $\sqrt{\dfrac{x}{x^2 - 2x - 35}}$ **64.** $\sqrt{\dfrac{x}{x^2 - 9}}$

In Exercises 65–70, solve the inequality. (Round your answers to two decimal places.)

65. $0.4x^2 + 5.26 < 10.2$

66. $-1.3x^2 + 3.78 > 2.12$

67. $-0.5x^2 + 12.5x + 1.6 > 0$

68. $1.2x^2 + 4.8x + 3.1 < 5.3$

69. $\dfrac{1}{2.3x - 5.2} > 3.4$ **70.** $\dfrac{2}{3.1x - 3.7} > 5.8$

HEIGHT OF A PROJECTILE In Exercises 71 and 72, use the position equation $s = -16t^2 + v_0t + s_0$, where s represents the height of an object (in feet), v_0 represents the initial velocity of the object (in feet per second), s_0 represents the initial height of the object (in feet), and t represents the time (in seconds).

71. A projectile is fired straight upward from ground level $(s_0 = 0)$ with an initial velocity of 160 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height exceed 384 feet?

72. A projectile is fired straight upward from ground level $(s_0 = 0)$ with an initial velocity of 128 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height be less than 128 feet?

73. GEOMETRY A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters. Within what bounds must the length of the rectangle lie?

74. GEOMETRY A rectangular parking lot with a perimeter of 440 feet is to have an area of at least 8000 square feet. Within what bounds must the length of the rectangle lie?

75. COST, REVENUE, AND PROFIT The revenue and cost equations for a product are $R = x(75 - 0.0005x)$ and $C = 30x + 250{,}000$, where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$750,000? What is the price per unit?

76. COST, REVENUE, AND PROFIT The revenue and cost equations for a product are

$$R = x(50 - 0.0002x) \quad \text{and} \quad C = 12x + 150{,}000$$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$1,650,000? What is the price per unit?

77. SCHOOL ENROLLMENT The numbers N (in millions) of students enrolled in schools in the United States from 1995 through 2006 are shown in the table. (Source: U.S. Census Bureau)

Year	Number, N
1995	69.8
1996	70.3
1997	72.0
1998	72.1
1999	72.4
2000	72.2
2001	73.1
2002	74.0
2003	74.9
2004	75.5
2005	75.8
2006	75.2

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a quartic model for the data.

(c) Graph the model and the scatter plot in the same viewing window. How well does the model fit the data?

(d) According to the model, during what range of years will the number of students enrolled in schools exceed 74 million?

(e) Is the model valid for long-term predictions of student enrollment in schools? Explain.

78. SAFE LOAD The maximum safe load uniformly distributed over a one-foot section of a two-inch-wide wooden beam is approximated by the model Load $= 168.5d^2 - 472.1$, where d is the depth of the beam.

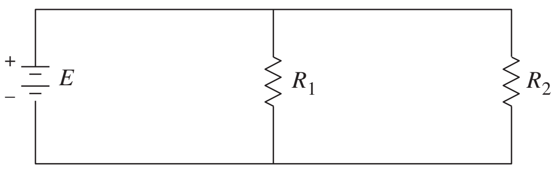
(a) Evaluate the model for $d = 4$, $d = 6$, $d = 8$, $d = 10$, and $d = 12$. Use the results to create a bar graph.

(b) Determine the minimum depth of the beam that will safely support a load of 2000 pounds.

79. RESISTORS When two resistors of resistances R_1 and R_2 are connected in parallel (see figure), the total resistance R satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Find R_1 for a parallel circuit in which $R_2 = 2$ ohms and R must be at least 1 ohm.



80. TEACHER SALARIES The mean salaries S (in thousands of dollars) of classroom teachers in the United States from 2000 through 2007 are shown in the table.

Year	Salary, S
2000	42.2
2001	43.7
2002	43.8
2003	45.0
2004	45.6
2005	45.9
2006	48.2
2007	49.3

A model that approximates these data is given by

$$S = \frac{42.6 - 1.95t}{1 - 0.06t}$$

where t represents the year, with $t = 0$ corresponding to 2000. (Source: Educational Research Service, Arlington, VA)

(a) Use a graphing utility to create a scatter plot of the data. Then graph the model in the same viewing window.

(b) How well does the model fit the data? Explain.

(c) According to the model, in what year will the salary for classroom teachers exceed $60,000?

(d) Is the model valid for long-term predictions of classroom teacher salaries? Explain.

EXPLORATION

TRUE OR FALSE? In Exercises 81 and 82, determine whether the statement is true or false. Justify your answer.

81. The zeros of the polynomial $x^3 - 2x^2 - 11x + 12 \geq 0$ divide the real number line into four test intervals.

82. The solution set of the inequality $\frac{3}{2}x^2 + 3x + 6 \geq 0$ is the entire set of real numbers.

In Exercises 83–86, (a) find the interval(s) for b such that the equation has at least one real solution and (b) write a conjecture about the interval(s) based on the values of the coefficients.

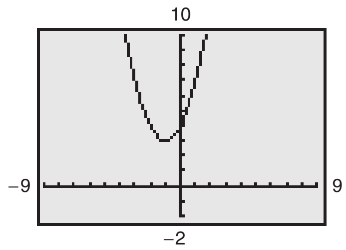
83. $x^2 + bx + 4 = 0$

84. $x^2 + bx - 4 = 0$

85. $3x^2 + bx + 10 = 0$

86. $2x^2 + bx + 5 = 0$

87. GRAPHICAL ANALYSIS You can use a graphing utility to verify the results in Example 4. For instance, the graph of $y = x^2 + 2x + 4$ is shown below. Notice that the y-values are greater than 0 for all values of x, as stated in Example 4(a). Use the graphing utility to graph $y = x^2 + 2x + 1$, $y = x^2 + 3x + 5$, and $y = x^2 - 4x + 4$. Explain how you can use the graphs to verify the results of parts (b), (c), and (d) of Example 4.



88. CAPSTONE Consider the polynomial

$(x - a)(x - b)$

and the real number line shown below.

(a) Identify the points on the line at which the polynomial is zero.

(b) In each of the three subintervals of the line, write the sign of each factor and the sign of the product.

(c) At what x-values does the polynomial change signs?

2 Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 2.1	Analyze graphs of quadratic functions *(p. 126).*	Let a, b,and c be real numbers with $a \neq 0$. The function given by $f(x) = ax^2 + bx + c$ is called a quadratic function. Its graph is a "U-shaped" curve called a parabola.	1, 2
	Write quadratic functions in standard form and use the results to sketch graphs of functions *(p. 129).*	The quadratic function $f(x) = a(x - h)^2 + k$, $a \neq 0$, is in standard form. The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is (h, k). If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward.	3–20
	Find minimum and maximum values of quadratic functions in real-life applications *(p. 131).*	Consider $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(\frac{b}{2a}\right)\right)$. If $a > 0$, f has a *minimum* at $x = -b/(2a)$. If $a < 0$, f has a *maximum* at $x = -b/(2a)$.	21–24
Section 2.2	Use transformations to sketch graphs of polynomial functions *(p. 136).*	The graph of a polynomial function is continuous (no breaks, holes, or gaps) and has only smooth, rounded turns.	25–30
	Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions *(p. 138).*	Consider the graph of $f(x) = a_nx^n + \cdot \cdot \cdot + a_1x + a_0$. **When *n* is odd:** If $a_n > 0$, the graph falls to the left and rises to the right. If $a_n < 0$, the graph rises to the left and falls to the right. **When *n* is even:** If $a_n > 0$, the graph rises to the left and right. If $a_n < 0$, the graph falls to the left and right.	31–34
	Find and use zeros of polynomial functions as sketching aids *(p. 139).*	If f is a polynomial function and a is a real number, the following are equivalent: (1) $x = a$ is a *zero* of f, (2) $x = a$ is a *solution* of the equation $f(x) = 0$, (3) $(x - a)$ is a *factor* of $f(x)$, and (4) $(a, 0)$ is an *x-intercept* of the graph of f.	35–44
	Use the Intermediate Value Theorem to help locate zeros of polynomial functions *(p. 143).*	Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.	45–48
Section 2.3	Use long division to divide polynomials by other polynomials *(p. 150).*	Dividend: $x^2 + 3x + 5$; Divisor: $x + 1$; Quotient: $x + 2$; Remainder: 3 $\dfrac{x^2 + 3x + 5}{x + 1} = x + 2 + \dfrac{3}{x + 1}$	49–54
	Use synthetic division to divide polynomials by binomials of the form $(x - k)$ *(p. 153).*	Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$ $\begin{array}{r\|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$ Remainder: 1 Quotient: $x^3 - 3x^2 - x + 1$	55–60
	Use the Remainder Theorem and the Factor Theorem *(p. 154).*	**The Remainder Theorem:** If a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$. **The Factor Theorem:** A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.	61–66
Section 2.4	Use the imaginary unit i to write complex numbers *(p. 159).*	If a and b are real numbers, $a + bi$ is a complex number. Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other if and only if $a = c$ and $b = d$.	67–70

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 2.4	Add, subtract, and multiply complex numbers *(p. 160)*.	**Sum:** $(a + bi) + (c + di) = (a + c) + (b + d)i$ **Difference:** $(a + bi) - (c + di) = (a - c) + (b - d)i$	71–78
	Use complex conjugates to write the quotient of two complex numbers in standard form *(p. 162)*.	The numbers $a + bi$ and $a - bi$ are complex conjugates. To write $(a + bi)/(c + di)$ in standard form, multiply the numerator and denominator by $c - di$.	79–82
	Find complex solutions of quadratic equations *(p. 163)*.	If a is a positive number, the principal square root of the negative number $-a$ is defined as $\sqrt{-a} = \sqrt{a}i$.	83–86
Section 2.5	Use the Fundamental Theorem of Algebra to find the number of zeros of polynomial functions *(p. 166)*.	**The Fundamental Theorem of Algebra** If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.	87–92
	Find rational zeros of polynomial functions *(p. 167)*, and conjugate pairs of complex zeros *(p. 170)*.	The Rational Zero Test relates the possible rational zeros of a polynomial to the leading coefficient and to the constant term of the polynomial. Let $f(x)$ be a polynomial function that has real coefficients. If $a + bi$ $(b \neq 0)$ is a zero of the function, the conjugate $a - bi$ is also a zero of the function.	93–102
	Find zeros of polynomials by factoring *(p. 170)*.	Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.	103–110
	Use Descartes's Rule of Signs *(p. 173)* and the Upper and Lower Bound Rules *(p. 174)* to find zeros of polynomials.	**Descartes's Rule of Signs** Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$. **1.** The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer. **2.** The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.	111–114
Section 2.6	Find the domains *(p. 181)*, and vertical and horizontal asymptotes *(p. 182)* of rational functions.	The domain of a rational function of x includes all real numbers except x-values that make the denominator zero. The line $x = a$ is a vertical asymptote of the graph of f if $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ as $x \rightarrow a$, either from the right or from the left. The line $y = b$ is a horizontal asymptote of the graph of f if $f(x) \rightarrow b$ as $x \rightarrow \infty$. or $x \rightarrow -\infty$.	115–122
	Analyze and sketch graphs of rational functions *(p. 184)* including functions with slant asymptotes *(p. 187)*.	Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, the graph of the function has a slant asymptote.	123–138
	Use rational functions to model and solve real-life problems *(p. 188)*.	A rational function can be used to model the cost of removing a given percent of smokestack pollutants at a utility company that burns coal. (See Example 8.)	139–142
Section 2.7	Solve polynomial *(p. 194)* and rational inequalities *(p. 198)*.	Use the concepts of key numbers and test intervals to solve both polynomial and rational inequalities.	143–150
	Use inequalities to model and solve real-life problems *(p. 199)*.	A common application of inequalities involves profit P, revenue R, and cost C. (See Example 6.)	151, 152

2 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

2.1 In Exercises 1 and 2, graph each function. Compare the graph of each function with the graph of $y = x^2$.

1. (a) $f(x) = 2x^2$
(b) $g(x) = -2x^2$
(c) $h(x) = x^2 + 2$
(d) $k(x) = (x + 2)^2$

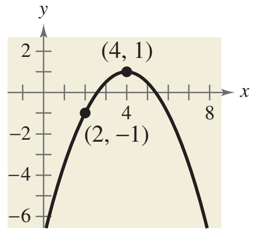
2. (a) $f(x) = x^2 - 4$
(b) $g(x) = 4 - x^2$
(c) $h(x) = (x - 3)^2$
(d) $k(x) = \frac{1}{2}x^2 - 1$

In Exercises 3–14, write the quadratic function in standard form and sketch its graph. Identify the vertex, axis of symmetry, and x-intercept(s).

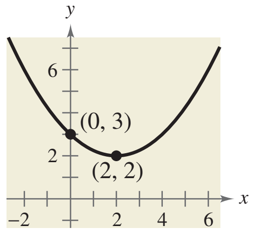
3. $g(x) = x^2 - 2x$
4. $f(x) = 6x - x^2$
5. $f(x) = x^2 + 8x + 10$
6. $h(x) = 3 + 4x - x^2$
7. $f(t) = -2t^2 + 4t + 1$
8. $f(x) = x^2 - 8x + 12$
9. $h(x) = 4x^2 + 4x + 13$
10. $f(x) = x^2 - 6x + 1$
11. $h(x) = x^2 + 5x - 4$
12. $f(x) = 4x^2 + 4x + 5$
13. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$
14. $f(x) = \frac{1}{2}(6x^2 - 24x + 22)$

In Exercises 15–20, write the standard form of the equation of the parabola that has the indicated vertex and whose graph passes through the given point.

15.



16.



17. Vertex: $(1, -4)$; point: $(2, -3)$
18. Vertex: $(2, 3)$; point: $(-1, 6)$
19. Vertex: $\left(-\frac{3}{2}, 0\right)$; point: $\left(-\frac{9}{2}, -\frac{11}{4}\right)$
20. Vertex: $(3, 3)$; point: $\left(\frac{1}{4}, \frac{4}{5}\right)$

21. GEOMETRY The perimeter of a rectangle is 1000 meters.

(a) Draw a diagram that gives a visual representation of the problem. Label the length and width as x and y, respectively.
(b) Write y as a function of x. Use the result to write the area as a function of x.
(c) Of all possible rectangles with perimeters of 1000 meters, find the dimensions of the one with the maximum area.

22. MAXIMUM REVENUE The total revenue R earned (in dollars) from producing a gift box of candles is given by

$$R(p) = -10p^2 + 800p$$

where p is the price per unit (in dollars).

(a) Find the revenues when the prices per box are \$20, \$25, and \$30.
(b) Find the unit price that will yield a maximum revenue. What is the maximum revenue? Explain your results.

23. MINIMUM COST A soft-drink manufacturer has daily production costs of

$$C = 70{,}000 - 120x + 0.055x^2$$

where C is the total cost (in dollars) and x is the number of units produced. How many units should be produced each day to yield a minimum cost?

24. SOCIOLOGY The average age of the groom at a first marriage for a given age of the bride can be approximated by the model

$$y = -0.107x^2 + 5.68x - 48.5, \quad 20 \le x \le 25$$

where y is the age of the groom and x is the age of the bride. Sketch a graph of the model. For what age of the bride is the average age of the groom 26? (Source: U.S. Census Bureau)

2.2 In Exercises 25–30, sketch the graphs of $y = x^n$ and the transformation.

25. $y = x^3, \quad f(x) = -(x - 2)^3$
26. $y = x^3, \quad f(x) = -4x^3$
27. $y = x^4, \quad f(x) = 6 - x^4$
28. $y = x^4, \quad f(x) = 2(x - 8)^4$
29. $y = x^5, \quad f(x) = (x - 5)^5$
30. $y = x^5, \quad f(x) = \frac{1}{2}x^5 + 3$

In Exercises 31–34, describe the right-hand and left-hand behavior of the graph of the polynomial function.

31. $f(x) = -2x^2 - 5x + 12$

32. $f(x) = \frac{1}{2}x^3 + 2x$

33. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

34. $h(x) = -x^7 + 8x^2 - 8x$

In Exercises 35–40, find all the real zeros of the polynomial function. Determine the multiplicity of each zero and the number of turning points of the graph of the function. Use a graphing utility to verify your answers.

35. $f(x) = 3x^2 + 20x - 32$
36. $f(x) = x(x + 3)^2$

37. $f(t) = t^3 - 3t$
38. $f(x) = x^3 - 8x^2$

39. $f(x) = -18x^3 + 12x^2$
40. $g(x) = x^4 + x^3 - 12x^2$

In Exercises 41–44, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

41. $f(x) = -x^3 + x^2 - 2$

42. $g(x) = 2x^3 + 4x^2$

43. $f(x) = x(x^3 + x^2 - 5x + 3)$

44. $h(x) = 3x^2 - x^4$

In Exercises 45–48, (a) use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. (b) Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of the graphing utility to verify your results.

45. $f(x) = 3x^3 - x^2 + 3$

46. $f(x) = 0.25x^3 - 3.65x + 6.12$

47. $f(x) = x^4 - 5x - 1$

48. $f(x) = 7x^4 + 3x^3 - 8x^2 + 2$

2.3 In Exercises 49–54, use long division to divide.

49. $\dfrac{30x^2 - 3x + 8}{5x - 3}$
50. $\dfrac{4x + 7}{3x - 2}$

51. $\dfrac{5x^3 - 21x^2 - 25x - 4}{x^2 - 5x - 1}$

52. $\dfrac{3x^4}{x^2 - 1}$

53. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

54. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

In Exercises 55–58, use synthetic division to divide.

55. $\dfrac{6x^4 - 4x^3 - 27x^2 + 18x}{x - 2}$
56. $\dfrac{0.1x^3 + 0.3x^2 - 0.5}{x - 5}$

57. $\dfrac{2x^3 - 25x^2 + 66x + 48}{x - 8}$

58. $\dfrac{5x^3 + 33x^2 + 50x - 8}{x + 4}$

In Exercises 59 and 60, use synthetic division to determine whether the given values of x are zeros of the function.

59. $f(x) = 20x^4 + 9x^3 - 14x^2 - 3x$

(a) $x = -1$ (b) $x = \frac{3}{4}$ (c) $x = 0$ (d) $x = 1$

60. $f(x) = 3x^3 - 8x^2 - 20x + 16$

(a) $x = 4$ (b) $x = -4$ (c) $x = \frac{2}{3}$ (d) $x = -1$

In Exercises 61 and 62, use the Remainder Theorem and synthetic division to find each function value.

61. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$

(a) $f(-3)$ (b) $f(-1)$

62. $g(t) = 2t^5 - 5t^4 - 8t + 20$

(a) $g(-4)$ (b) $g(\sqrt{2})$

In Exercises 63–66, (a) verify the given factor(s) of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

	Function	*Factor(s)*
63.	$f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
64.	$f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
65.	$f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2)(x - 3)$
66.	$f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2)(x - 5)$

2.4 In Exercises 67–70, write the complex number in standard form.

67. $8 + \sqrt{-100}$
68. $5 - \sqrt{-49}$

69. $i^2 + 3i$
70. $-5i + i^2$

In Exercises 71–78, perform the operation and write the result in standard form.

71. $(7 + 5i) + (-4 + 2i)$

72. $\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right) - \left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)$

73. $7i(11 - 9i)$
74. $(1 + 6i)(5 - 2i)$

75. $(10 - 8i)(2 - 3i)$
76. $i(6 + i)(3 - 2i)$

77. $(8 - 5i)^2$

78. $(4 + 7i)^2 + (4 - 7i)^2$

In Exercises 79 and 80, write the quotient in standard form.

79. $\dfrac{6+i}{4-i}$ **80.** $\dfrac{8-5i}{i}$

In Exercises 81 and 82, perform the operation and write the result in standard form.

81. $\dfrac{4}{2-3i}+\dfrac{2}{1+i}$ **82.** $\dfrac{1}{2+i}-\dfrac{5}{1+4i}$

In Exercises 83–86, find all solutions of the equation.

83. $5x^2+2=0$ **84.** $2+8x^2=0$

85. $x^2-2x+10=0$ **86.** $6x^2+3x+27=0$

2.5 In Exercises 87–92, find all the zeros of the function.

87. $f(x)=4x(x-3)^2$

88. $f(x)=(x-4)(x+9)^2$

89. $f(x)=x^2-11x+18$

90. $f(x)=x^3+10x$

91. $f(x)=(x+4)(x-6)(x-2i)(x+2i)$

92. $f(x)=(x-8)(x-5)^2(x-3+i)(x-3-i)$

In Exercises 93 and 94, use the Rational Zero Test to list all possible rational zeros of *f*.

93. $f(x)=-4x^3+8x^2-3x+15$

94. $f(x)=3x^4+4x^3-5x^2-8$

In Exercises 95–100, find all the rational zeros of the function.

95. $f(x)=x^3+3x^2-28x-60$

96. $f(x)=4x^3-27x^2+11x+42$

97. $f(x)=x^3-10x^2+17x-8$

98. $f(x)=x^3+9x^2+24x+20$

99. $f(x)=x^4+x^3-11x^2+x-12$

100. $f(x)=25x^4+25x^3-154x^2-4x+24$

In Exercises 101 and 102, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

101. $\frac{2}{3}, 4, \sqrt{3}i$ **102.** $2, -3, 1-2i$

In Exercises 103–106, use the given zero to find all the zeros of the function.

	Function	*Zero*
103.	$f(x)=x^3-4x^2+x-4$	i
104.	$h(x)=-x^3+2x^2-16x+32$	$-4i$
105.	$g(x)=2x^4-3x^3-13x^2+37x-15$	$2+i$
106.	$f(x)=4x^4-11x^3+14x^2-6x$	$1-i$

In Exercises 107–110, find all the zeros of the function and write the polynomial as a product of linear factors.

107. $f(x)=x^3+4x^2-5x$

108. $g(x)=x^3-7x^2+36$

109. $g(x)=x^4+4x^3-3x^2+40x+208$

110. $f(x)=x^4+8x^3+8x^2-72x-153$

In Exercises 111 and 112, use Descartes's Rule of Signs to determine the possible numbers of positive and negative zeros of the function.

111. $g(x)=5x^3+3x^2-6x+9$

112. $h(x)=-2x^5+4x^3-2x^2+5$

In Exercises 113 and 114, use synthetic division to verify the upper and lower bounds of the real zeros of *f*.

113. $f(x)=4x^3-3x^2+4x-3$

(a) Upper: $x=1$ (b) Lower: $x=-\frac{1}{4}$

114. $f(x)=2x^3-5x^2-14x+8$

(a) Upper: $x=8$ (b) Lower: $x=-4$

2.6 In Exercises 115–118, find the domain of the rational function.

115. $f(x)=\dfrac{3x}{x+10}$ **116.** $f(x)=\dfrac{4x^3}{2+5x}$

117. $f(x)=\dfrac{8}{x^2-10x+24}$ **118.** $f(x)=\dfrac{x^2+x-2}{x^2+4}$

In Exercises 119–122, identify any vertical or horizontal asymptotes.

119. $f(x)=\dfrac{4}{x+3}$ **120.** $f(x)=\dfrac{2x^2+5x-3}{x^2+2}$

121. $h(x)=\dfrac{5x+20}{x^2-2x-24}$ **122.** $h(x)=\dfrac{x^3-4x^2}{x^2+3x+2}$

In Exercises 123–134, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical and horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

123. $f(x)=\dfrac{-3}{2x^2}$ **124.** $f(x)=\dfrac{4}{x}$

125. $g(x)=\dfrac{2+x}{1-x}$ **126.** $h(x)=\dfrac{x-4}{x-7}$

127. $p(x)=\dfrac{5x^2}{4x^2+1}$ **128.** $f(x)=\dfrac{2x}{x^2+4}$

129. $f(x)=\dfrac{x}{x^2+1}$ **130.** $h(x)=\dfrac{9}{(x-3)^2}$

131. $f(x) = \dfrac{-6x^2}{x^2 + 1}$

132. $f(x) = \dfrac{2x^2}{x^2 - 4}$

133. $f(x) = \dfrac{6x^2 - 11x + 3}{3x^2 - x}$

134. $f(x) = \dfrac{6x^2 - 7x + 2}{4x^2 - 1}$

In Exercises 135–138, (a) state the domain of the function, (b) identify all intercepts, (c) identify any vertical and slant asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

135. $f(x) = \dfrac{2x^3}{x^2 + 1}$

136. $f(x) = \dfrac{x^2 + 1}{x + 1}$

137. $f(x) = \dfrac{3x^3 - 2x^2 - 3x + 2}{3x^2 - x - 4}$

138. $f(x) = \dfrac{3x^3 - 4x^2 - 12x + 16}{3x^2 + 5x - 2}$

139. **AVERAGE COST** A business has a production cost of $C = 0.5x + 500$ for producing x units of a product. The average cost per unit, $\overline{C}$, is given by

$$\overline{C} = \frac{C}{x} = \frac{0.5x + 500}{x}, \quad x > 0.$$

Determine the average cost per unit as x increases without bound. (Find the horizontal asymptote.)

140. **SEIZURE OF ILLEGAL DRUGS** The cost C (in millions of dollars) for the federal government to seize $p\%$ of an illegal drug as it enters the country is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of seizing 25%, 50%, and 75% of the drug.

(c) According to this model, would it be possible to seize 100% of the drug?

141. **PAGE DESIGN** A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are 2 inches deep and the margins on each side are 2 inches wide.

(a) Draw a diagram that gives a visual representation of the problem.

(b) Write a function for the total area A of the page in terms of x.

(c) Determine the domain of the function based on the physical constraints of the problem.

(d) Use a graphing utility to graph the area function and approximate the page size for which the least amount of paper will be used. Verify your answer numerically using the *table* feature of the graphing utility.

142. **PHOTOSYNTHESIS** The amount y of CO_2 uptake (in milligrams per square decimeter per hour) at optimal temperatures and with the natural supply of CO_2 is approximated by the model

$$y = \frac{18.47x - 2.96}{0.23x + 1}, \quad x > 0$$

where x is the light intensity (in watts per square meter). Use a graphing utility to graph the function and determine the limiting amount of CO_2 uptake.

2.7 In Exercises 143–150, solve the inequality.

143. $12x^2 + 5x < 2$

144. $3x^2 + x \ge 24$

145. $x^3 - 16x \ge 0$

146. $12x^3 - 20x^2 < 0$

147. $\dfrac{2}{x + 1} \le \dfrac{3}{x - 1}$

148. $\dfrac{x - 5}{3 - x} < 0$

149. $\dfrac{x^2 - 9x + 20}{x} \le 0$

150. $\dfrac{1}{x - 2} > \dfrac{1}{x}$

151. **INVESTMENT** P dollars invested at interest rate r compounded annually increases to an amount

$$A = P(1 + r)^2$$

in 2 years. An investment of \$5000 is to increase to an amount greater than \$5500 in 2 years. The interest rate must be greater than what percent?

152. **POPULATION OF A SPECIES** A biologist introduces 200 ladybugs into a crop field. The population P of the ladybugs is approximated by the model

$$P = \frac{1000(1 + 3t)}{5 + t}$$

where t is the time in days. Find the time required for the population to increase to at least 2000 ladybugs.

EXPLORATION

TRUE OR FALSE? In Exercises 153 and 154, determine whether the statement is true or false. Justify your answer.

153. A fourth-degree polynomial with real coefficients can have -5, $-8i$, $4i$, and 5 as its zeros.

154. The domain of a rational function can never be the set of all real numbers.

155. **WRITING** Explain how to determine the maximum or minimum value of a quadratic function.

156. **WRITING** Explain the connections among factors of a polynomial, zeros of a polynomial function, and solutions of a polynomial equation.

157. **WRITING** Describe what is meant by an asymptote of a graph.

2 CHAPTER TEST

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

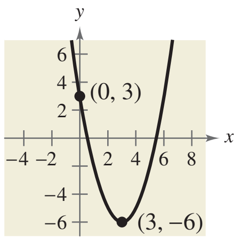


FIGURE FOR 2

1. Describe how the graph of g differs from the graph of $f(x) = x^2$.
   (a) $g(x) = 2 - x^2$ (b) $g(x) = \left(x - \frac{3}{2}\right)^2$
2. Find an equation of the parabola shown in the figure at the left.
3. The path of a ball is given by $y = -\frac{1}{20}x^2 + 3x + 5$, where y is the height (in feet) of the ball and x is the horizontal distance (in feet) from where the ball was thrown.
   (a) Find the maximum height of the ball.
   (b) Which number determines the height at which the ball was thrown? Does changing this value change the coordinates of the maximum height of the ball? Explain.
4. Determine the right-hand and left-hand behavior of the graph of the function $h(t) = -\frac{3}{4}t^5 + 2t^2$. Then sketch its graph.
5. Divide using long division.
   $$\frac{3x^3 + 4x - 1}{x^2 + 1}$$
6. Divide using synthetic division.
   $$\frac{2x^4 - 5x^2 - 3}{x - 2}$$
7. Use synthetic division to show that $x = \frac{5}{2}$ is a zero of the function given by
   $f(x) = 2x^3 - 5x^2 - 6x + 15.$
   Use the result to factor the polynomial function completely and list all the real zeros of the function.
8. Perform each operation and write the result in standard form.
   (a) $10i - \left(3 + \sqrt{-25}\right)$ (b) $\left(2 + \sqrt{3}i\right)\left(2 - \sqrt{3}i\right)$
9. Write the quotient in standard form: $\dfrac{5}{2 + i}$.

In Exercises 10 and 11, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

10. $0, 3, 2 + i$
11. $1 - \sqrt{3}i, 2, 2$

In Exercises 12 and 13, find all the zeros of the function.

12. $f(x) = 3x^3 + 14x^2 - 7x - 10$
13. $f(x) = x^4 - 9x^2 - 22x - 24$

In Exercises 14–16, identify any intercepts and asymptotes of the graph of the function. Then sketch a graph of the function.

14. $h(x) = \dfrac{4}{x^2} - 1$
15. $f(x) = \dfrac{2x^2 - 5x - 12}{x^2 - 16}$
16. $g(x) = \dfrac{x^2 + 2}{x - 1}$

In Exercises 17 and 18, solve the inequality. Sketch the solution set on the real number line.

17. $2x^2 + 5x > 12$
18. $\dfrac{2}{x} \le \dfrac{1}{x + 6}$