5.5 Buckingham Pi theorem

The second step is in a dimensional analysis is to make dimensionless groups. That task is simpler by knowing in advance how many groups to look for. The Buckingham Pi theorem provides that number. I derive it with a series of examples.

Here is a possible beginning of the theorem statement: *The number of dimensionless groups is. . . .* Try it on the light-bending example. How many groups can the variables θ, G, m, r, and c produce? The possibilities include θ, θ^2, Gm/rc^2, $\theta Gm/rc^2$, and so on. The possibilities are infinite! Now apply the theorem statement to estimating the size of hydrogen, before including quantum mechanics in the list of variables. That list is a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e. That list produces no dimensionless groups. So it seems that the number of groups would be zero – if no groups are possible – or infinity, if even one group is possible.

Here is an improved theorem statement taking account of the redundancy: *The number of* ***independent*** *dimensionless groups is. . . .* To complete the statement, try a few examples:

1. Bending of light. The five quantities θ, G, m, r, and c produce two independent groups. A convenient choice for the two groups is θ and Gm/rc^2, but any other independent set is equally valid, even if not as intuitive.
2. Size of hydrogen without quantum mechanics. The three quantities a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e produce zero groups.
3. Size of hydrogen with quantum mechanics. The four quantities a_0 (the size), $e^2/4\pi\epsilon_0$, m_e, and $\hbar$ produce one independent group.

These examples fit a simple pattern:

$$\text{no. of independent groups} = \text{no. of quantities} - 3.$$

The 3 is a bit distressing because it is a magic number with no explanation. It is also the number of basic dimensions: length, mass, and time. So perhaps the statement is

$$\text{no. of independent groups} = \text{no. of quantities} - \text{no. of dimensions}.$$

Test this statement with additional examples:

1. Period of a spring–mass system. The quantities are T (the period), k, m, and x_0 (the amplitude). These four quantities form one independent dimensionless group, which could be kT^2/m. This result is consistent with the proposed theorem.
2. Period of a spring–mass system (without x_0). Since the amplitude x_0 does not affect the period, the quantities could have been T (the period), k, and m. These three quantities form one independent dimensionless group, which again could be kT^2/m. This result is also consistent with the proposed theorem, since T, k, and m contain only two dimensions (mass and time).

The theorem is safe until we try to derive Newton's second law. The force F depends on mass m and acceleration a. Those three quantities contain three dimensions – mass, length, and time. Three minus three is zero, so the proposed theorem predicts zero independent dimensionless groups. Whereas $F = ma$ tells me that F/ma is a dimensionless group.

This problem can be fixed by adding one word to the statement. Look at the dimensions of F, m, and a. All the dimensions – M or MLT^{-2} or LT^{-2} – can be constructed from only *two* dimensions: M and LT^{-2}. The key idea is that the original set of three dimensions are not independent, whereas the pair M and LT^{-2} are independent. So:

Var	Dim	What
F	MLT^{-2}	force
m	M	mass
a	LT^{-2}	acceleration

$$\text{\# of independent groups} = \text{\# of quantities} - \text{\# of } \textit{independent} \text{ dimensions}.$$

That statement is the Buckingham Pi theorem [3].

5.6 Drag

For the final example of dimensional analysis, we revisit the cone experiment of Section 3.3.1. That analysis, using conservation, concluded that the drag force on the cones is given by

$$F_{\text{drag}} \sim \rho v^2 A, \tag{5.1}$$

where ρ is the density of the fluid (e.g. air or water), v is the speed of the cone, and A is its cross-sectional area. What can dimensional analysis tell us about this problem?

The strategy is to find the quantities that affect F_{drag}, find their dimensions, and then find dimensionless groups.

▶ *On what quantities does the drag depend, and what are their dimensions?*

The drag force depends on four quantities: two parameters of the cone and two parameters of the fluid (air). Any dimensionless form can be built from dimensionless groups: from dimensionless products of the variables. Because any equation describing the world can be written in a dimensionless form, and any dimensionless form can be written using dimensionless groups, any equation describing the world can be written using dimensionless groups.

v	speed of the cone	LT^{-1}
r	size of the cone	L
ρ	density of air	ML^{-3}
ν	viscosity of air	L^2T^{-1}

Problem 5.3 Kepler's third law
Use dimensional analysis to derive Kepler's third law connecting the orbital period of a planet to its orbital radius (for a circular orbit).

▶ *What dimensionless groups can be constructed for the drag problem?*

According to the Buckingham Pi theorem, the five quantities and three independent dimensions give rise to two independent dimensionless groups. One dimensionless group could be $F/\rho v^2 r^2$. A second group could be rv/ν. Any other dimensionless group can be constructed from these two groups (Problem 5.4), so the problem is indeed described by two independent dimensionless groups. The most general dimensionless statement is then

$$\text{one group} = f(\text{second group}), \tag{5.2}$$

where f is a still-unknown (but dimensionless) function.

▶ *Which dimensionless group belongs on the left side?*

The goal is to synthesize a formula for F, and F appears only in the first group $F/\rho v^2 r^2$. With that constraint in mind, place the first group on the left side rather than wrapping it in the still-mysterious function f. With this choice, the most general statement about drag force is

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right). \tag{5.3}$$

The physics of the (steady-state) drag force on the cone is all contained in the dimensionless function f. However, dimensional analysis cannot tell us anything about that function. To make progress requires incorporating new knowledge. It can come from an experiment such as dropping the small and large cones, from wind-tunnel tests at various speeds, and even from physical reasoning.

Before reexamining the results of the cone experiment in dimensionless form, let's name the two dimensionless groups. The first one, $F/\rho v^2 r^2$, is traditionally written in a slightly different form:

$$\frac{F}{\frac{1}{2}\rho v^2 A}, \tag{5.4}$$

where A is the cross-sectional area of the cone. The $1/2$ is an arbitrary choice, but it is the usual choice: It is convenient and is reminiscent of the $1/2$ in the kinetic energy formula $mv^2/2$. Written in that way, the first dimensionless group is called the drag coefficient and is abbreviated c_{d}. The second group, rv/ν, is called the Reynolds number. It is traditionally written in terms of the diameter rather than the radius:

$$\frac{vL}{\nu}, \tag{5.5}$$

where L is the diameter of the object.

The conclusion of the dimensional analysis is then

$$\text{drag coefficient} = f(\text{Reynolds number}). \tag{5.6}$$

Now let's see how the cone experiment fits into this dimensionless framework. The experimental data was that the small and large cones fell at the same speed – roughly $1\,\mathrm{m\,s^{-1}}$. The conclusion is that the drag force is proportional to the cross-sectional area A. Because the drag coefficient is proportional to F/A, which is the same for the small and large cones, the small and large cones have the same drag coefficient.

Their Reynolds numbers, however, are not the same. For the small cone, the diameter is $2\,\mathrm{in} \times 0.75$ (why?), which is roughly $4\,\mathrm{cm}$. The Reynolds number is

$$\mathrm{Re} \sim \frac{1\,\mathrm{m\,s^{-1}} \times 0.04\,\mathrm{m}}{1.5 \cdot 10^{-5}\,\mathrm{m^2\,s^{-1}}}, \tag{5.7}$$

where $1\,\mathrm{m\,s^{-1}}$ is the fall speed and $1.5{\cdot}10^{-5}\,\mathrm{m^2\,s^{-1}}$ is the kinematic viscosity of air. Numerically, $\mathrm{Re}_{\text{small}} \sim 2000$. For the large cone, the fall speed and viscosity are the same as for the small cone, but the diameter is twice as large, so $\mathrm{Re}_{\text{large}} \sim 4000$. The result of the cone experiment is, in dimensionless form, that the drag coefficient is independent of Reynolds number – at least, for Reynolds numbers between 2000 and 4000.

This conclusion is valid for diverse shapes. The most extensive data on drag coefficient versus Reynolds number is for a sphere. That data is plotted logarithmically below (from *Fluid-dynamic Drag: Practical Information on Aerodynamic Drag and Hydrodynamic Resistance* by Sighard F. Hoerner):

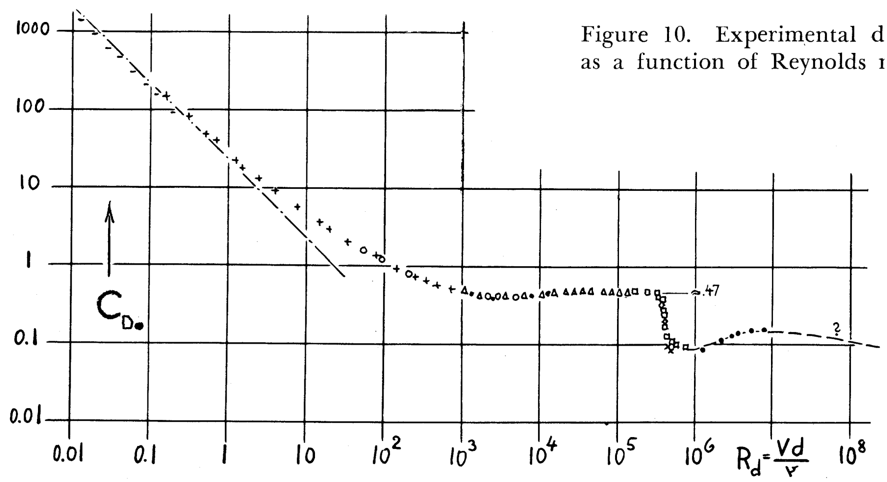

Figure 10. Experimental d
as a function of Reynolds ı

Just like the cones, the sphere's drag coefficient is almost constant in the Reynolds number range 2000 to 4000. This full graph has interesting features. First, toward the low-Reynolds-number end, the drag coefficient increases. Second, for high Reynolds numbers, the drag coefficient stays roughly constant until $\mathrm{Re} \sim 10^6$, where it rapidly drops by almost a factor of 5. The behavior at low Reynolds number will be explained in the chapter on easy (extreme) cases (Chapter 6). The drop in the drag coefficient, which relates to why golf balls have dimples, will be explained in the chapter on lumping (Chapter 7).

Problem 5.4 Only two groups
Show that F, v, r, ρ, and ν produce only two independent dimensionless groups.

Problem 5.5 Counting dimensionless groups
How many independent dimensionless groups are there in the following sets of variables:

a. size of hydrogen including relativistic effects:

$$e^2/4\pi\epsilon_0,\ \hbar,\ c,\ a_0 \text{ (Bohr radius)},\ m_e \text{ (electron mass)}.$$

b. period of a spring–mass system in a gravitational field:

$$T \text{ (period)},\ k \text{ (spring constant)},\ m,\ x_0 \text{ (amplitude)},\ g.$$

c. speed at which a free-falling object hits the ground:

$$v,\ g,\ h \text{ (initial drop height)}.$$

d. [tricky!] weight W of an object:

$$W,\ g,\ m.$$

Problem 5.6 Integrals by dimensions
You can use dimensions to do integrals. As an example, try this integral:

$$I(\beta) = \int_{-\infty}^{\infty} e^{-\beta x^2}\, dx.$$

Which choice has correct dimensions: (a.) $\sqrt{\pi}\beta^{-1}$ (b.) $\sqrt{\pi}\beta^{-1/2}$ (c.) $\sqrt{\pi}\beta^{1/2}$ (d.) $\sqrt{\pi}\beta^{1}$

Hints:

1. The dimensions of dx are the same as the dimensions of x.
2. Pick interesting dimensions for x, such as length. (If x is dimensionless then you cannot use dimensional analysis on the integral.)

Problem 5.7 How to avoid remembering lots of constants
Many atomic problems, such as the size or binding energy of hydrogen, end up in expressions with $\hbar$, the electron mass m_e, and $e^2/4\pi\epsilon_0$, which is a nicer way to express the squared electron charge. You can avoid having to remember those constants if instead you remember these values instead:

$$\hbar c \approx 200\,\text{eV nm} = 2000\,\text{eV Å}$$
$$m_e c^2 \sim 0.5 \cdot 10^6\,\text{eV}$$
$$\frac{e^2/4\pi\epsilon_0}{\hbar c} \equiv \alpha \approx \frac{1}{137} \quad \text{(fine-structure constant)}.$$

Use those values to evaluate the Bohr radius in angstroms (1 Å = 0.1 nm):

$$a_0 = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

As an example calculation using the $\hbar c$ value, here is the energy of a photon:

$$E = hf = 2\pi\hbar f = 2\pi\hbar\frac{c}{\lambda},$$

where f is its frequency and λ is its wavelength. For green light, $\lambda \sim 600\,\text{nm}$, so

$$E \sim \frac{\overbrace{6}^{2\pi} \times \overbrace{200\,\text{eV nm}}^{\hbar c}}{\underbrace{600\,\text{nm}}_{\lambda}} \sim 2\,\text{eV}.$$

Problem 5.8 Heavy nuclei

In lecture we analyzed hydrogen, which is one electron bound to one proton. In this problem you study the innermost electron in an atom such as uranium that has many protons, and analyze one physical consequence of its binding energy.

So, imagine a nucleus with Z protons around which orbits one electron. Let $E(Z)$ be the binding energy (the hydrogen energy is the case $Z = 1$).

a. Show that the ratio $E(Z)/E(1)$ is Z^2.

b. In lecture, we derived that $E(1)$ is the kinetic energy of an electron moving with speed αc where α is the fine-structure constant (roughly 10^{-2}). How fast does the innermost electron move around a heavy nucleus with charge Z?

c. When that speed is comparable to the speed of light, the electron has a kinetic energy comparable to its (relativistic) rest energy. One consequence of such a high kinetic energy is that the electron has enough kinetic energy to produce a positron (an anti-electron) out of nowhere ('pair creation'). That positron leaves the nucleus, turning a proton into a neutron as it exits. So the atomic number Z drops by one: The nucleus is unstable! Relativity sets an upper limit for Z.

   Estimate that maximum Z and compare it with the Z for the heaviest stable nucleus (uranium).

Problem 5.9 Power radiated by an accelerating charge

Electromagnetism, where the usual derivations are so cumbersome, is an excellent area to apply dimensional analysis. In this problem you work out the power radiated by an accelerating charge, which is how radio stations work.

So, consider a particle with charge q, with position x, velocity v, and acceleration a. What variables are relevant to the radiated power P? The position cannot matter because it depends on the origin of the coordinate system, whereas the power radiated cannot depend on the origin. The velocity cannot matter because of relativity: You can transform to a reference frame where $v = 0$, but that change will not affect the radiation (otherwise you could distinguish a moving frame from a non-moving frame, in violation of the principle of relativity). So the acceleration a is all that's left to determine the radiated power. [This line of argument is slightly dodgy, but it works for low speeds.]

a. Using P, $q^2/4\pi\epsilon_0$, and a, how many dimensionless quantities can you form?

b. Fix the problem in the previous part by adding one quantity to the list of variables, and give a physical reason for including the quantity.

c. With the new list, use dimensionless groups to find the power radiated by an accelerating point charge. In case you are curious, the exact result contains a dimensionless factor of 2/3; dimensional analysis triumphs again!

Problem 5.10 Yield from an atomic bomb

Geoffrey Taylor, a famous Cambridge fluid mechanic, annoyed the US government by doing the following analysis. The question he answered: 'What was the yield (in kilotons of TNT) of the first atomic blast (in the New Mexico desert in 1945)?' Declassified pictures, which even had a scale bar, gave the following data on the radius of the explosion at various times:

t (ms)	R (m)
3.26	59.0
4.61	67.3
15.0	106.5
62.0	185.0

a. Use dimensional analysis to work out the relation between radius R, time t, blast energy E, and air density ρ.

b. Use the data in the table to estimate the blast energy E (in Joules).

c. Convert that energy to kilotons of TNT. One gram of TNT releases 1 kcal or roughly 4 kJ.

The actual value was 20 kilotons, a classified number when Taylor published his result ['The Formation of a Blast Wave by a Very Intense Explosion. II. The Atomic Explosion of 1945.', *Proceedings of the Royal Society of London. Series A, Mathematical and Physical* **201**(1065): 175–186 (22 March 1950)]

Problem 5.11 Atomic blast: A physical interpretation

Use energy densities and sound speeds to make a rough physical explanation of the result in the 'yield from an atomic bomb' problem.

Problem 5.12 Rolling down the plane

Four objects, made of identical steel, roll down an inclined plane without slipping. The objects are:

1. a large spherical shell,
2. a large disc,
3. a small solid sphere,
4. a small ring.

The large objects have three times the radius of the small objects. Rank the objects by their acceleration (highest acceleration first).

Check your results with exact calculation or with a home experiment.

Problem 5.13 Blackbody radiation

A hot object – a so-called blackbody – radiates energy, and the flux F depends on the temperature T. In this problem you derive the connection using dimensional analysis. The goal is to find F as a function of T. But you need more quantities.

a. What are the dimensions of flux?

b. What two constants of nature should be included because blackbody radiation depends on the quantum theory of radiation?

c. What constant of nature should be included because you are dealing with temperature?

d. After doing the preceding parts, you have five variables. Explain why these five variables produce one dimensionless group, and use that fact to deduce the relation between flux and temperature.

e. Look up the Stefan–Boltzmann law and compare your result to it.